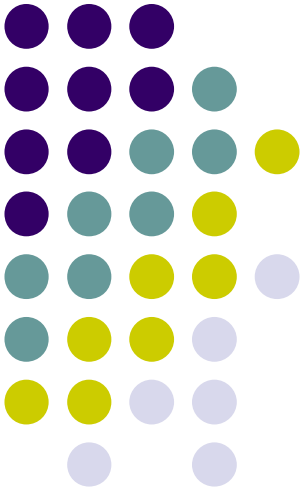
Deflection: Virtual Work Method; Beams and Frames

Theory of Structure - I



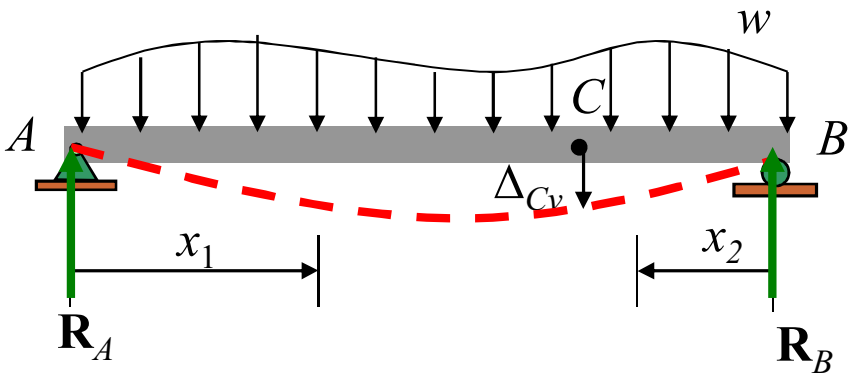
Contents

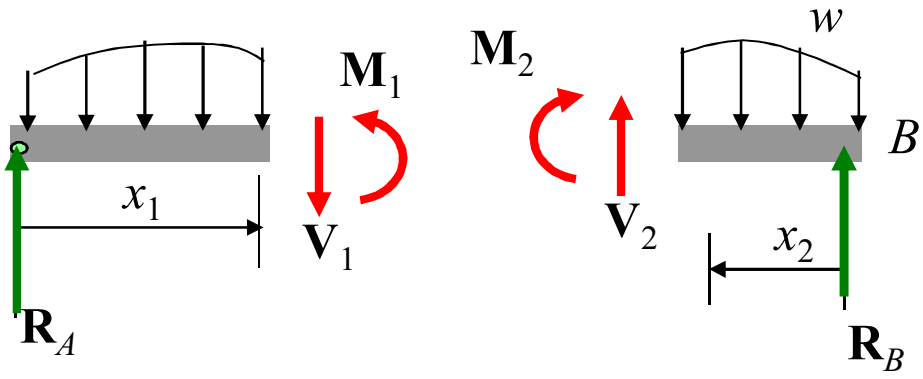
- Method of Virtual Work:
  - Beams
  - Frames

Method of Virtual Work : Beams and Frames

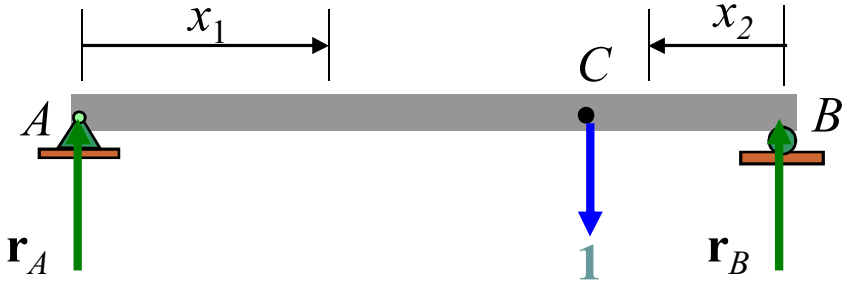
- *Vertical Displacement*

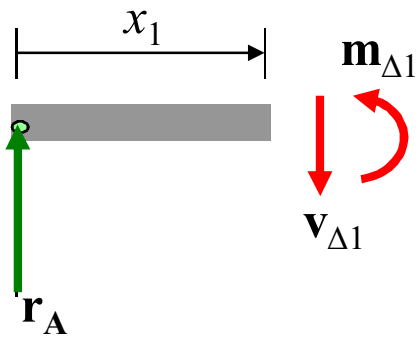
Real load

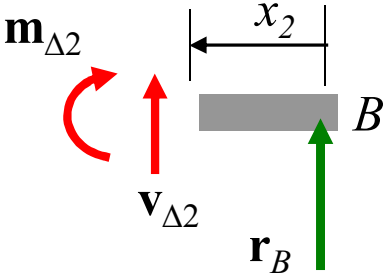




***Virtual* unit load**







$$1 \bullet \Delta_{Cv} = \int_0^L \frac{m_\Delta M}{EI} dx$$

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University of Engineering and
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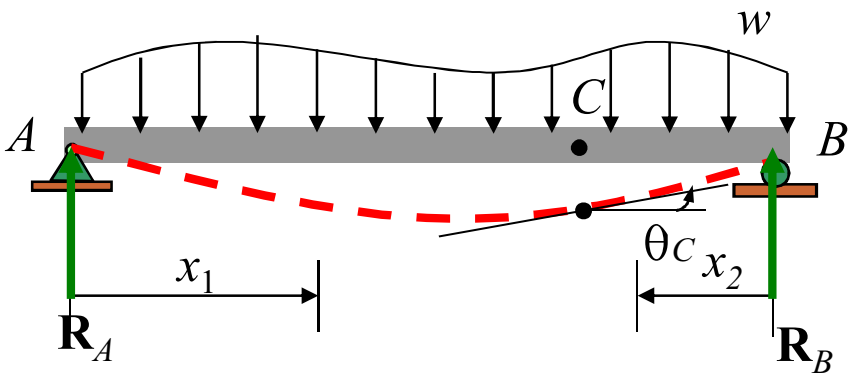
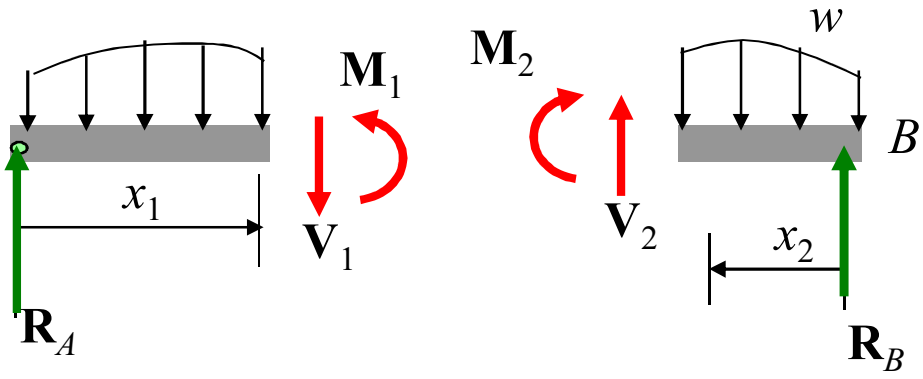
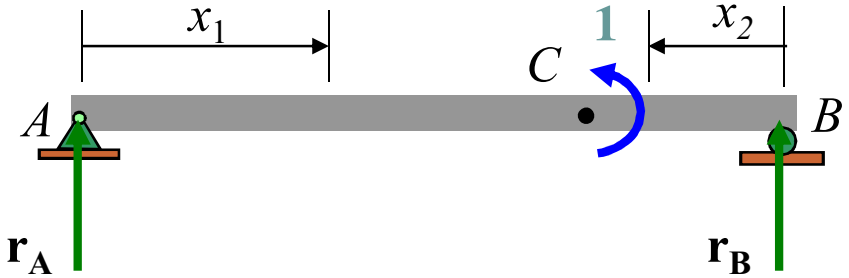
• ***Slope***

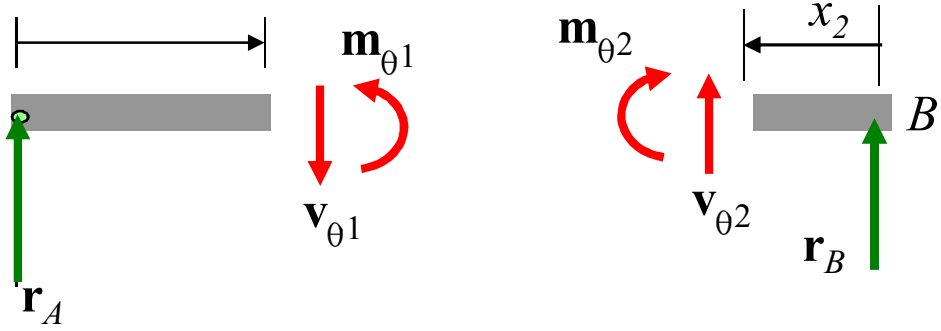
Real load





Virtual unit couple

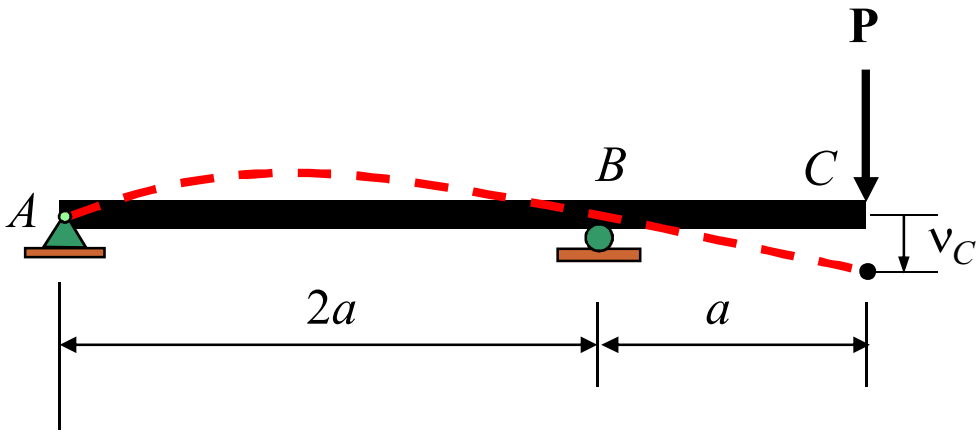




$$1 \bullet \theta_C = \int_0^L \frac{m_\theta M}{EI} dx$$

Example 8-18

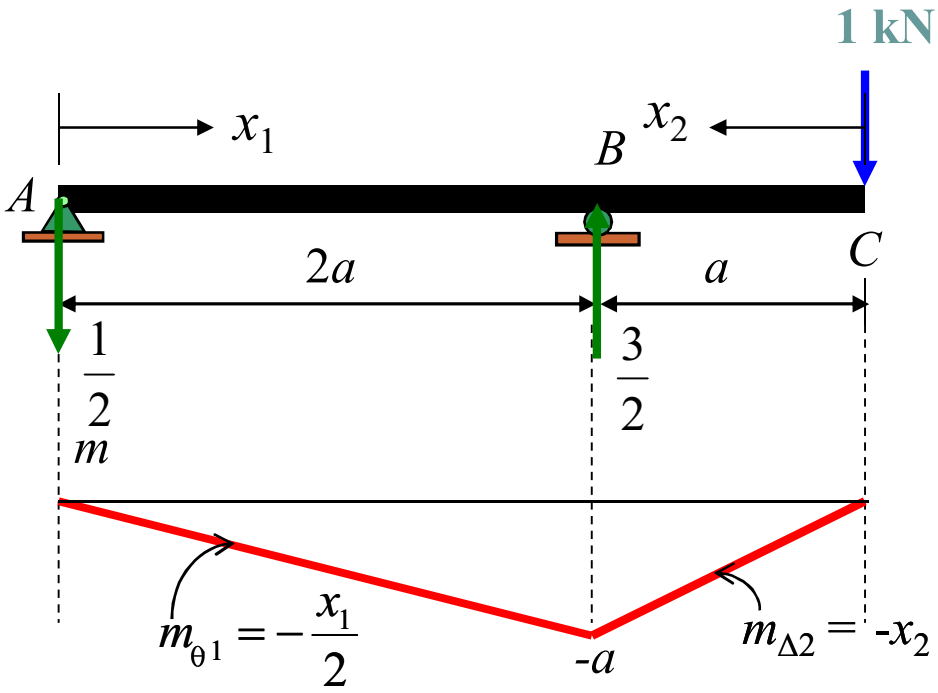
The beam shown is subjected to a load **P** at its end. Determine the slope and displacement at *C*. *EI* is constant.



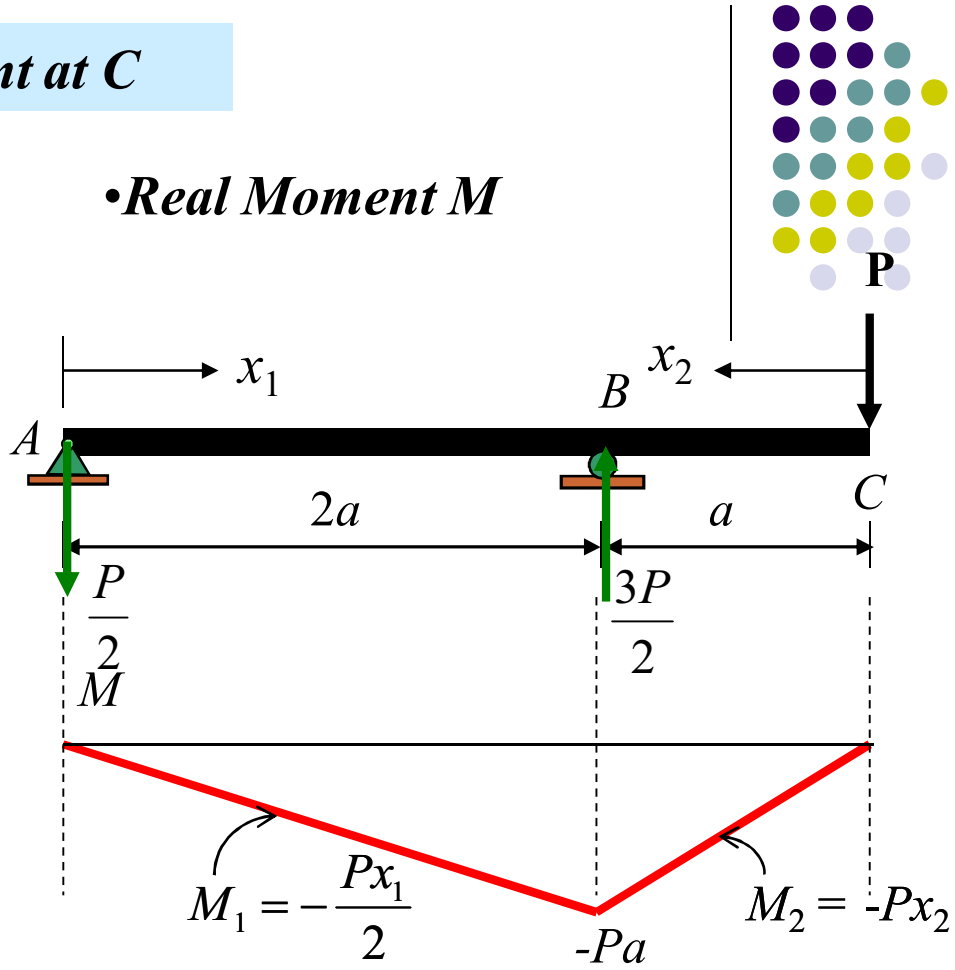
SOLUTION

Displacement at C

- ***Virtual Moment m_Δ***



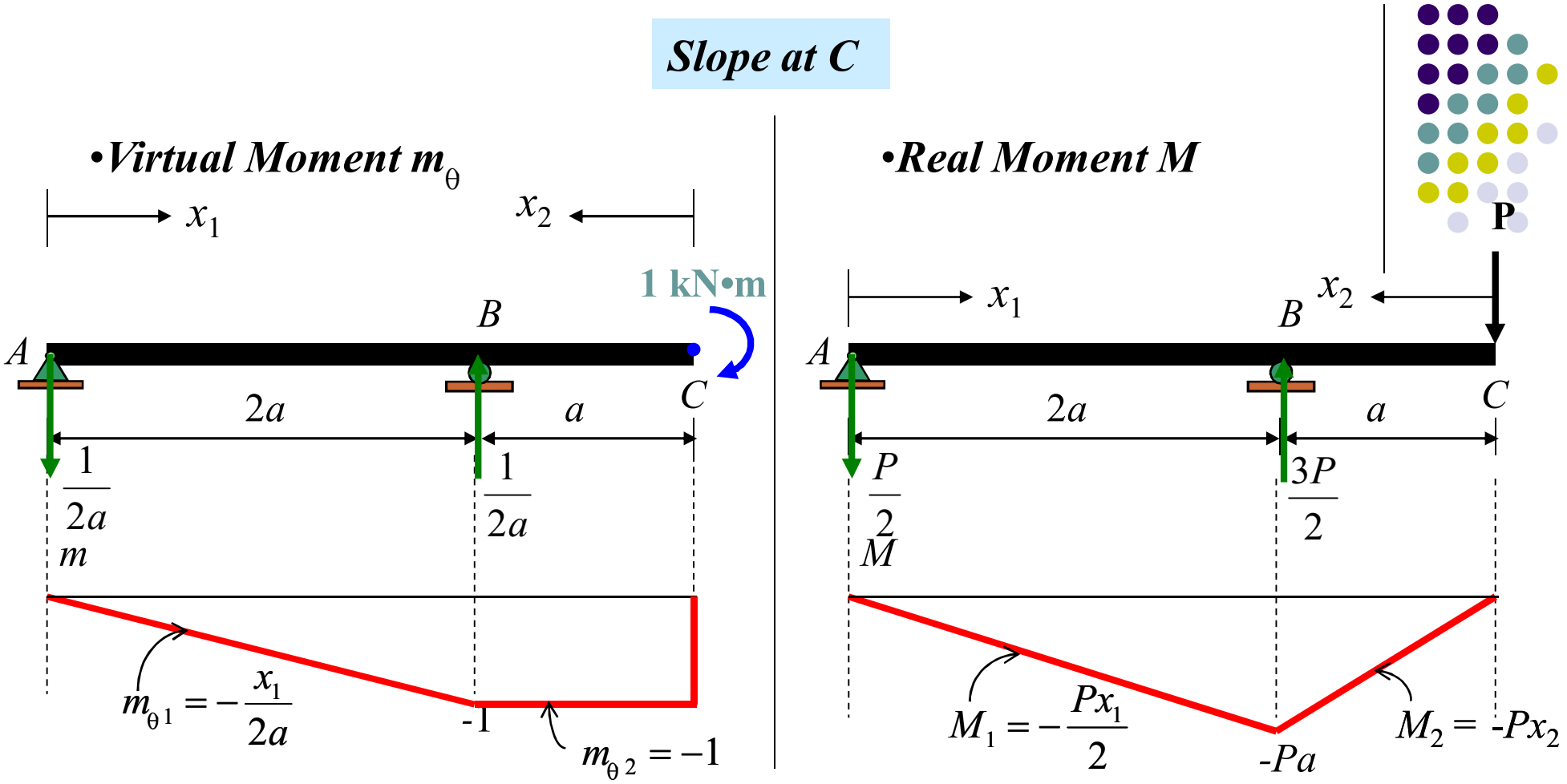
- ***Real Moment M***



$$(1kN)(\Delta_C) = \int_0^L \frac{m_\Delta M}{EI}\,dx = \frac{1}{EI}\int_0^{2a}(-\frac{x_1}{2})(-\frac{Px_1}{2})dx_1 + \frac{1}{EI}\int_0^{a}(-x_2)(-Px_2)dx_2$$

$$\Delta_C = (\frac{1}{EI})(\frac{P}{4})(\frac{x_1^3}{3})\Big|_0^{2a} + (\frac{P}{EI})(\frac{x_2^3}{3})\Big|_0^{a} = (\frac{1}{EI})(\frac{P}{4})(\frac{8a^3}{3}) + (\frac{P}{EI})(\frac{a^3}{3}) = \frac{Pa^3}{EI}, \quad \downarrow$$

Slope at C

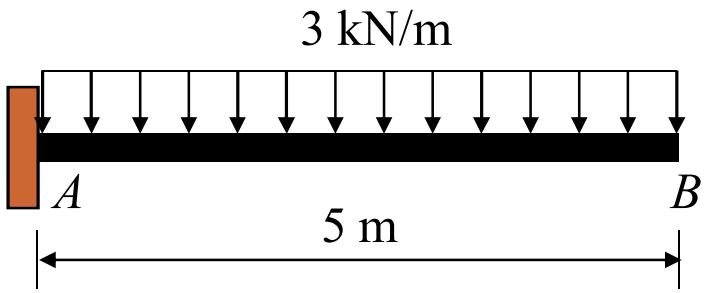


$$(1kN \bullet m)(\theta_C) = \int_0^L \frac{m_\theta M}{EI}\,dx = \frac{1}{EI}\int_0^{2a}(-\frac{x_1}{2a})(-\frac{Px_1}{2})dx_1 + \frac{1}{EI}\int_0^{a}(-1)(-Px_2)dx_2$$

$$\theta_C = (\frac{1}{EI})(\frac{P}{4a})(\frac{x_1^3}{3})\Big|_0^{2a} + (\frac{1}{EI})(\frac{Px_2^2}{2})\Big|_0^{a} = (\frac{1}{EI})(\frac{P}{4a})(\frac{8a^3}{3}) + (\frac{1}{EI})(\frac{Pa^2}{2}) = \frac{7}{6}(\frac{Pa^2}{EI}),$$

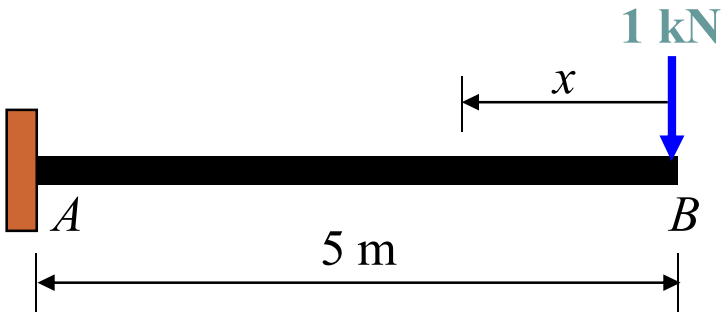
Example 8-19

Determine the slope and displacement of point B of the steel beam shown in the figure below. Take $E = 200$ GPa, $I = 250(10^6)$ mm^4.

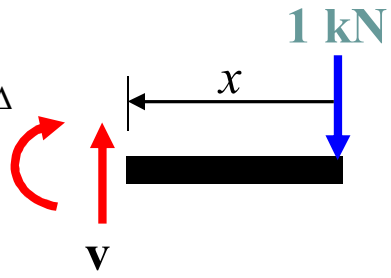


SOLUTION ***Vertical Displacement at B***

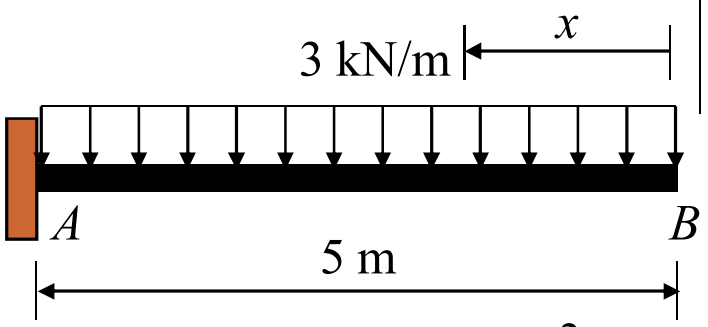
- ***Virtual Moment*** m_Δ



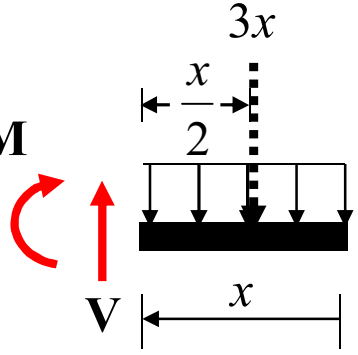
$-1x =$ $\mathbf{m}_\Delta$



- ***Real Moment M***



$-\frac{3x^2}{2} =$ **M**



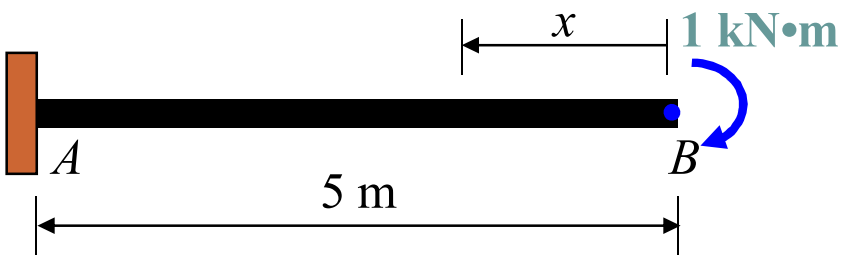
$$(1kN)(\Delta_B) = \int_0^L \frac{m_\Delta M}{EI}dx = \frac{1}{EI}\int_0^5 (-x)(-\frac{3x^2}{2})dx = \frac{1}{EI}\int_0^5 \frac{3x^3}{2} = \frac{1}{EI}(\frac{3x^4}{8}\Big|_0^5) = \frac{234.375kN^2 \bullet m^3}{EI}$$

$$\Delta_B = \frac{234.375kN \bullet m^3}{(200\times10^6 \frac{kN}{m})(250\times10^{-6}m^4)} = 0.00469\text{ m} = 4.69\text{mm}, \downarrow$$

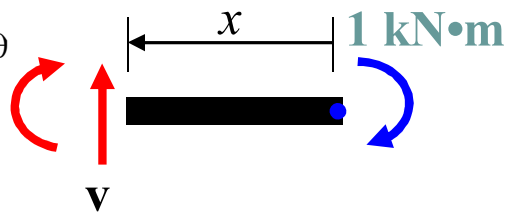
SOLUTION

Slope at B

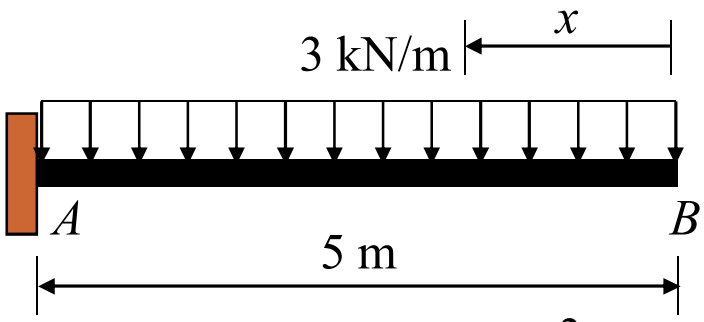
- ***Virtual Moment m_θ***



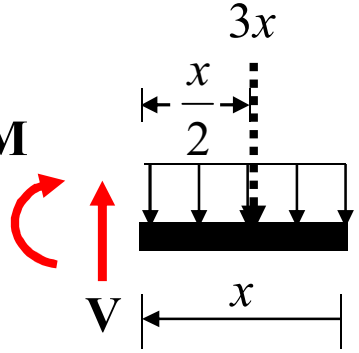
-1 = $\mathbf{m}_\theta$



- ***Real Moment M***



$-\frac{3x^2}{2} = \mathbf{M}$



$$(1kN \bullet m)(\theta_B) = \int_0^L \frac{m_\theta M}{EI} dx = \frac{1}{EI}\int_0^5 (-1)(-\frac{3x^2}{2})dx = \frac{1}{EI}\int_0^5 \frac{3x^2}{2} = \frac{1}{EI}(\frac{3x^3}{6}\Big|_0^5) = \frac{62.5kN^2 \bullet m^3}{EI}$$

$$\theta_B = \frac{62.5kN \bullet m^2}{(200\times10^6 \frac{kN}{m})(250\times10^{-6} m^4)} = 0.00125 \text{ rad},$$

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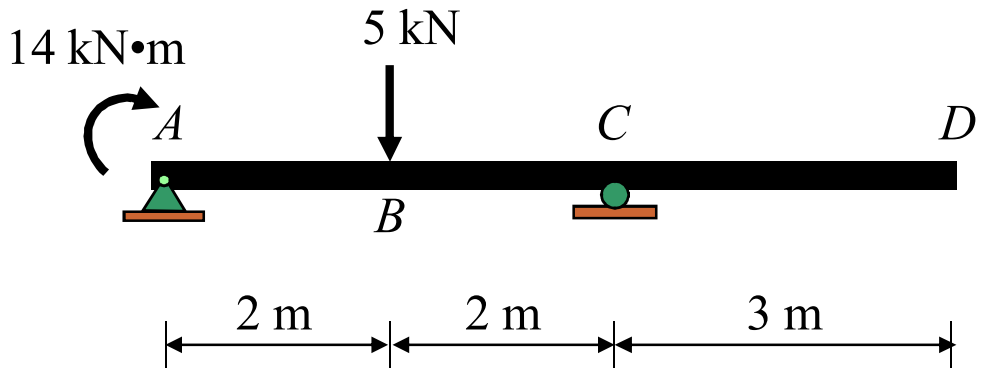
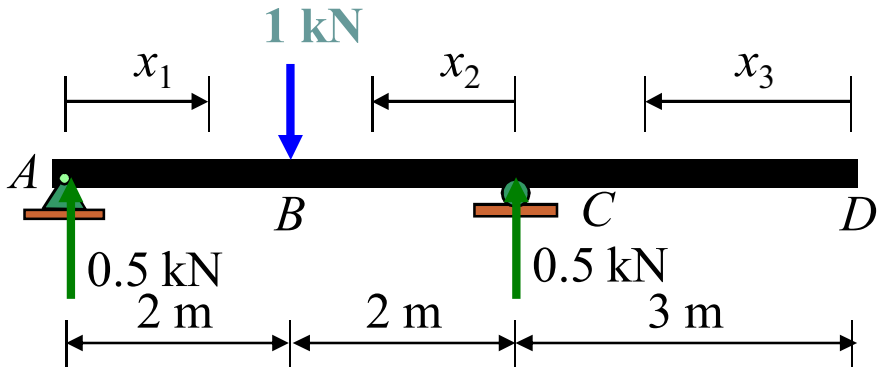
Example 8-20

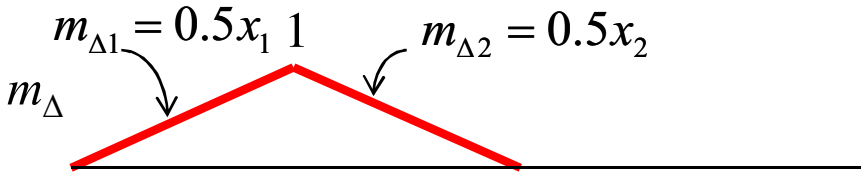
Determine the slope and displacement of point B of the steel beam shown in the figure below. Take $E = 200$ GPa, $I = 60(10^6)$ mm^4.



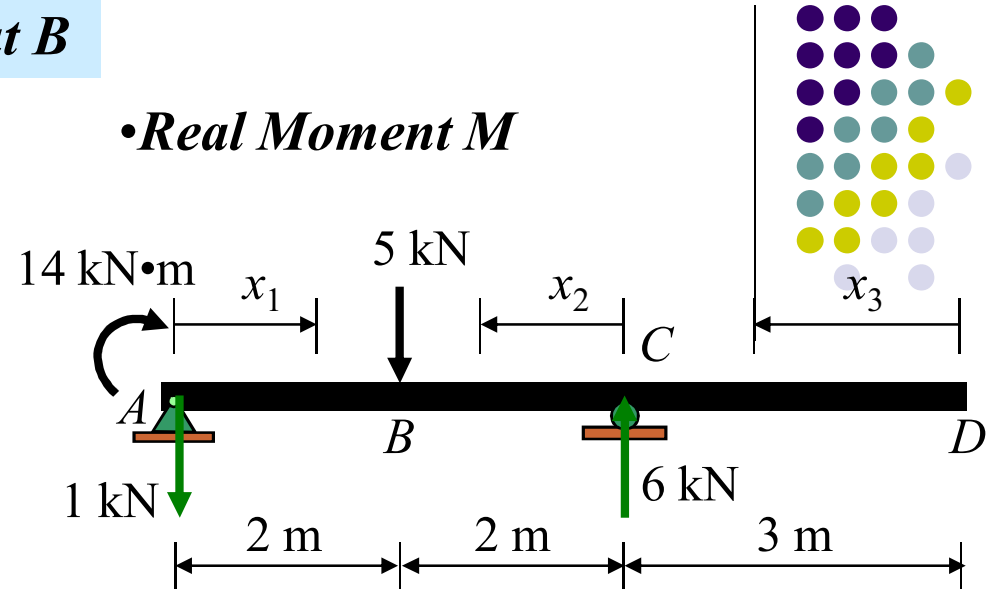
Displacement at B

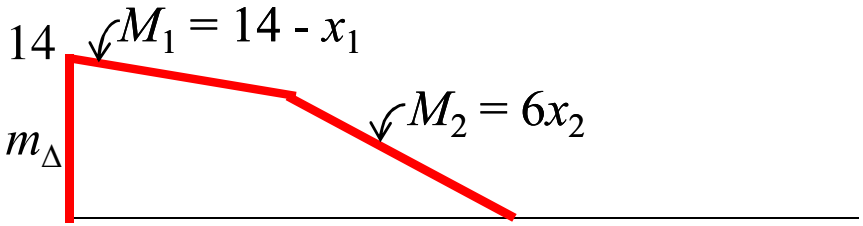
•*Virtual Moment* m_Δ





•*Real Moment M*





$$(1kN)(\Delta_B) = \int_0^L \frac{m_\Delta M}{EI}\,dx$$

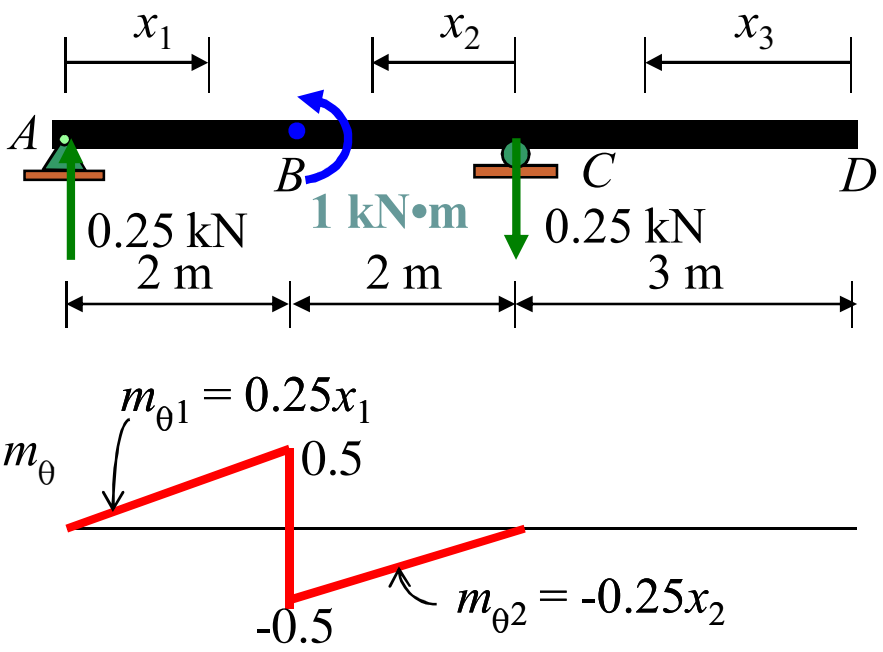
$$= \frac{1}{EI}\int_0^2 (0.5x_1)(14 - x_1)\,dx_1 + \frac{1}{EI}\int_0^2 (0.5x_2)(6x_2)\,dx_2 + \int_0^3 (0)(0)\,dx_3$$

$$= \frac{1}{EI}\int_0^2 (7x_1 - 0.5x_1^{\,2})\,dx_1 + \frac{1}{EI}\int_0^2 (3x_2^{\,2})\,dx_2 = \left(\frac{1}{EI}\right)\left(\frac{7x_1^{\,2}}{2} - \frac{0.5x_1^{\,3}}{3}\right)\Bigg|_0^2 + \left(\frac{1}{EI}\right)\left(\frac{3x_2^{\,3}}{3}\right)\Bigg|_0^2$$

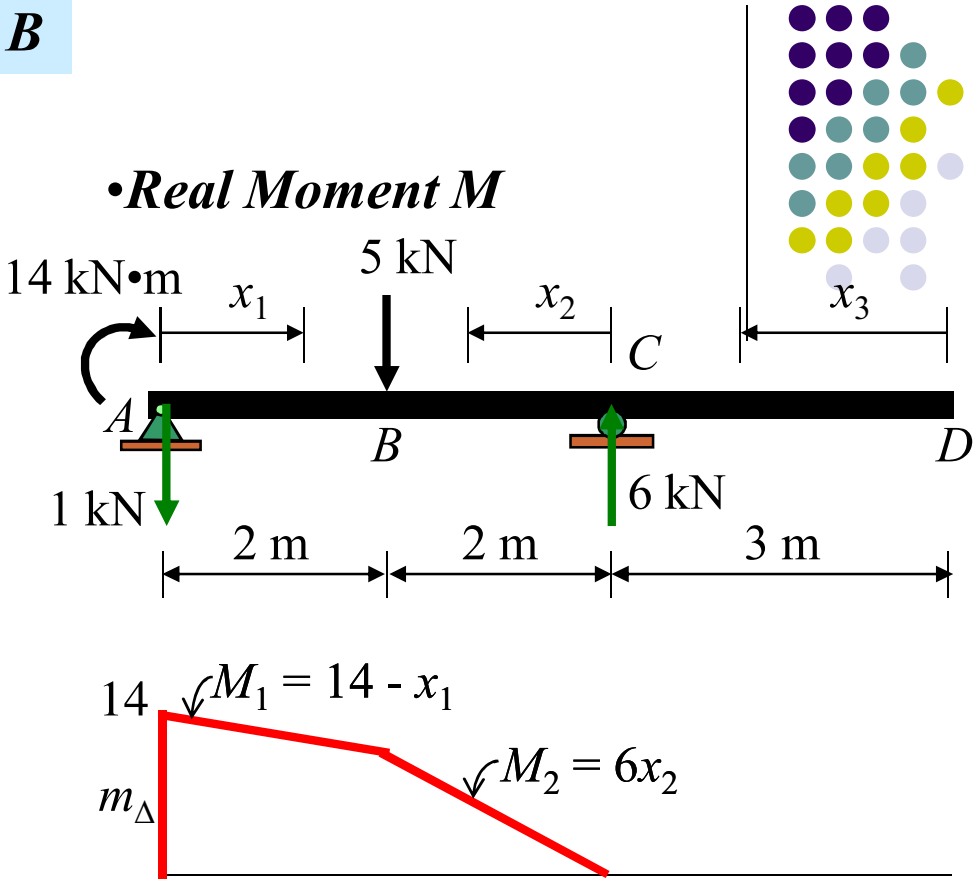
$$\Delta_B = \frac{20.667}{EI} = \frac{20.667}{(200)(60)} = 0.00172\ \text{m} = 1.72\ \text{mm},\ \downarrow$$

Slope at B

•*Virtual Moment* m_θ



•*Real Moment M*



$$(1kN \bullet m)(\theta_B) = \int_0^L \frac{m_\theta M}{EI} dx = \frac{1}{EI}\int_0^2 (0.25x_1)(14 - x_1)dx_1 + \frac{1}{EI}\int_0^2 (-0.25x_2)(6x_2)dx_2 + \int_0^3 (0)(0)dx_3$$

$$= \frac{1}{EI}\int_0^2 (3.5x_1 - 0.25x_1^{\,2})dx_1 + \frac{1}{EI}\int_0^2 (-1.5x_2^{\,2})dx_2$$

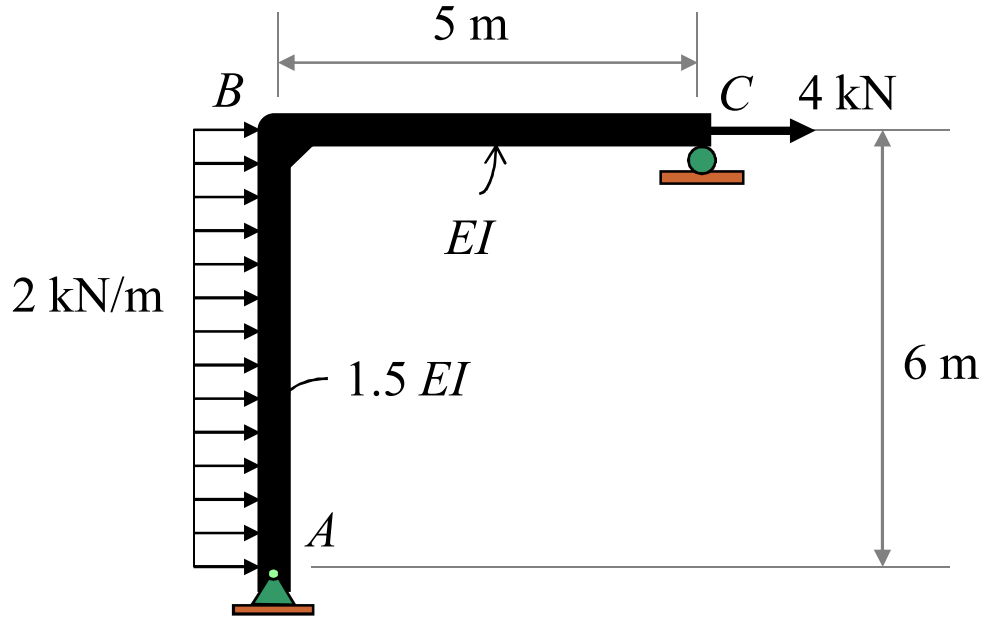
$$= \frac{1}{EI}\left(\frac{3.5x_1^{\,2}}{2} - \frac{0.25x_1^{\,3}}{3}\right)\Bigg|_0^2 + \frac{1}{EI}\left(-\frac{1.5x_2^{\,3}}{3}\right)\Bigg|_0^2$$

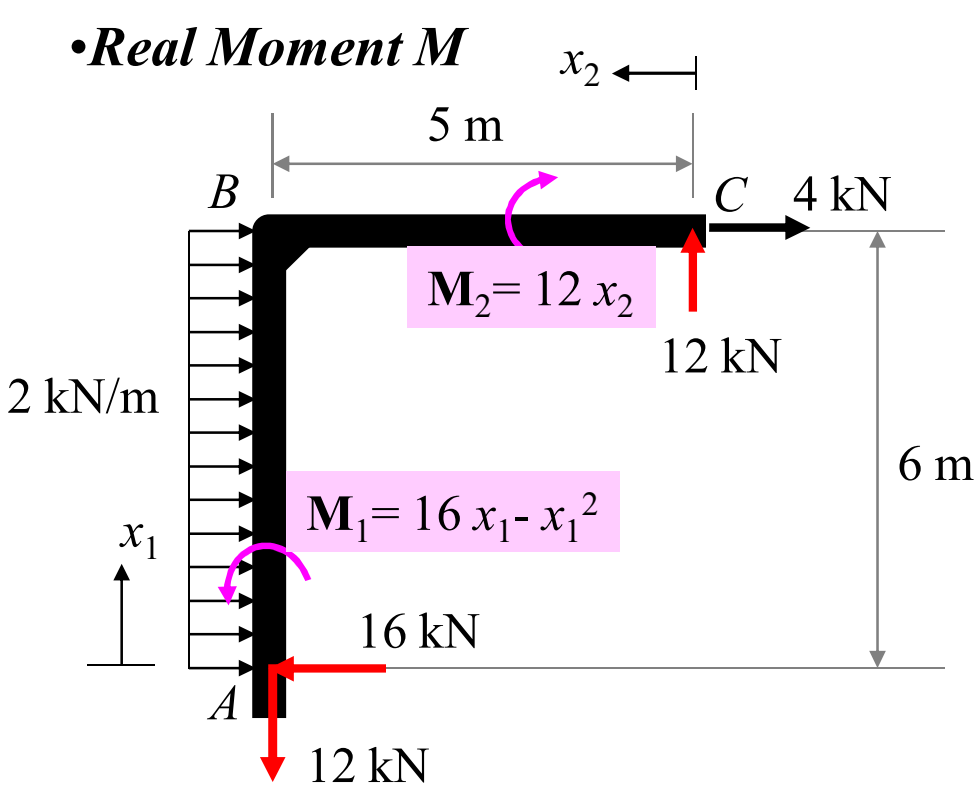
$$\theta_B = \frac{2.333}{EI} = \frac{2.333}{(200)(60)} = 0.000194 \text{ rad}$$

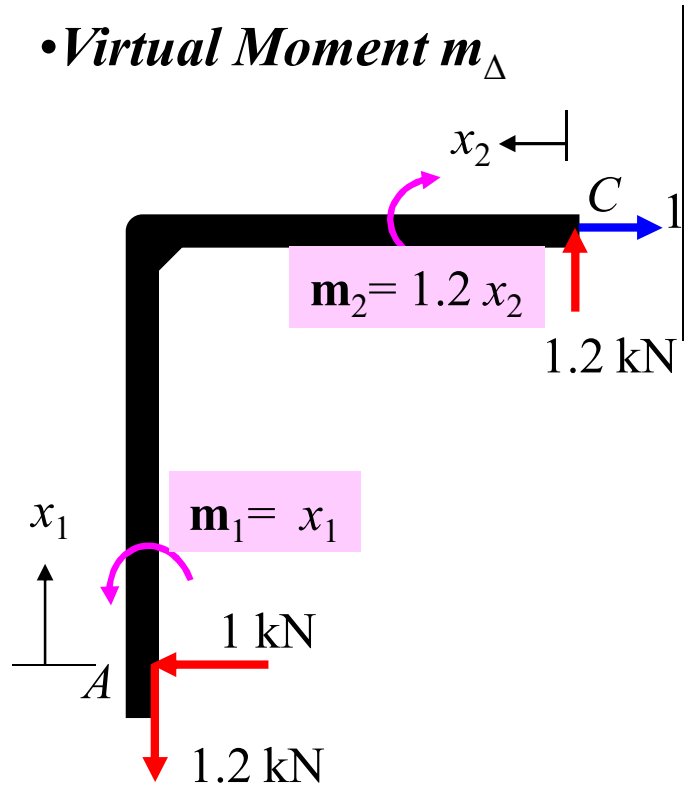
Example 8-21

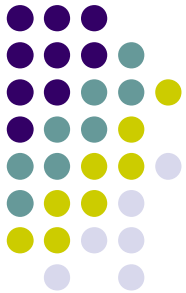
(a) Determine the slope and the horizontal displacement of point ***C*** on the frame.
(b) Draw the bending moment diagram and deflected curve.

$E = 200$ GPa
$I = 200(10^6)\ \text{mm}^4$





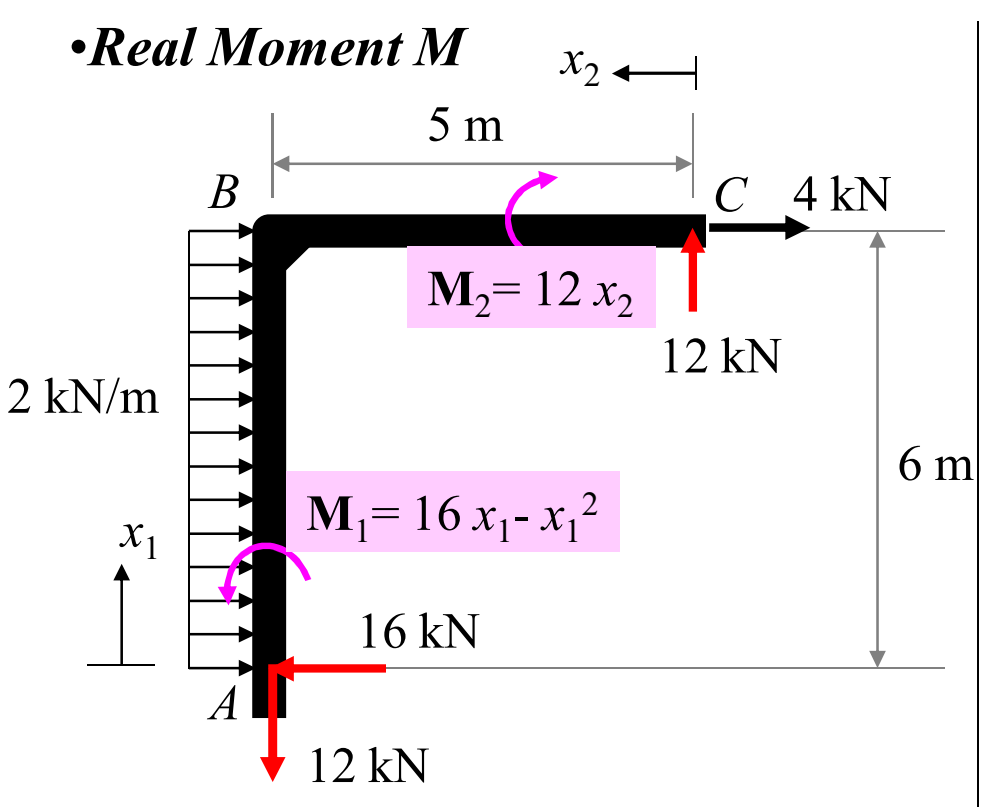


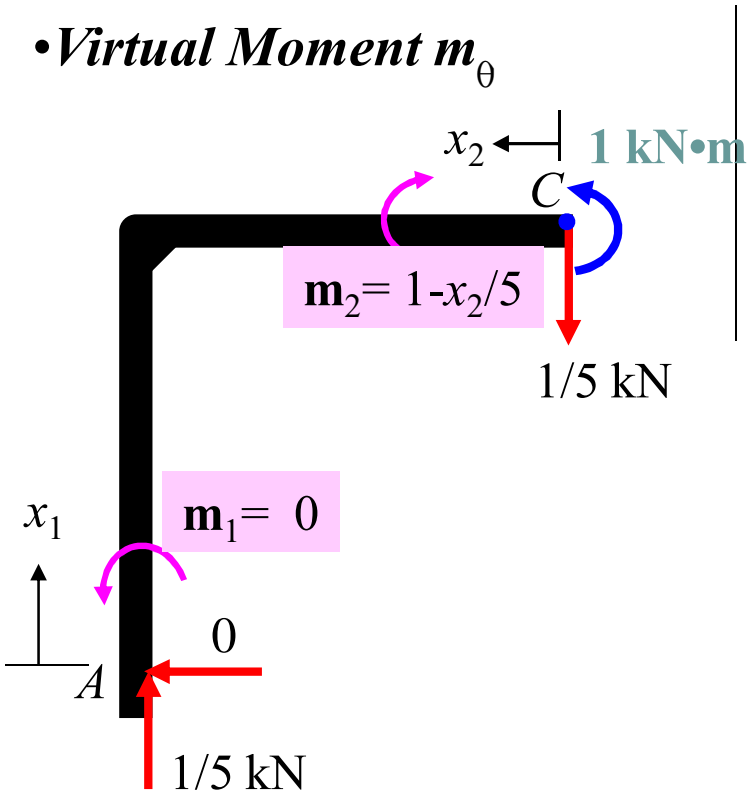


$$(1kN)(\Delta_{CH}) = \int_0^L \frac{mM}{EI}\,dx = \frac{1}{1.5EI}\int_0^6 (x_1)(16x_1 - x_1^{\,2})dx_1 + \frac{1}{EI}\int_0^5 (1.2x_2)(12x_2)dx_2$$

$$= \frac{1}{1.5EI}\int_0^6 (16x_1^{\,2} - x_1^{\,3})dx_1 + \frac{1}{EI}\int_0^5 (14.4x_2^{\,2})dx_2$$

$$\Delta_{CH} = \frac{1}{1.5EI}\left(\frac{16x_1^{\,3}}{3} - \frac{x_1^{\,4}}{4}\right)\Bigg|_0^6 + \frac{1}{EI}\left(\frac{14.4x_2^{\,3}}{3}\right)\Bigg|_0^5 = \frac{552}{EI} + \frac{600}{EI} = \frac{1152}{(200)(200)} = +28.8\ \text{mm}\ , \longrightarrow$$

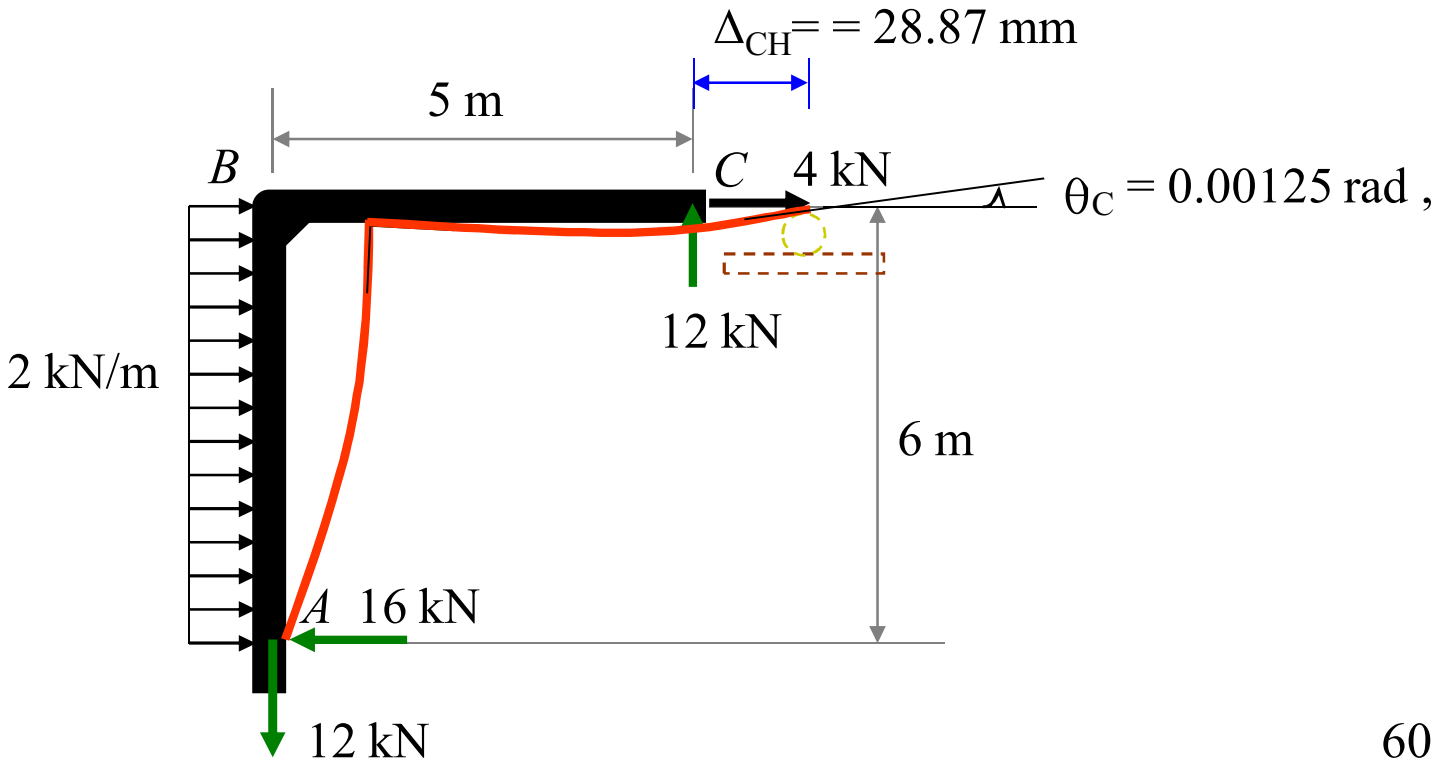


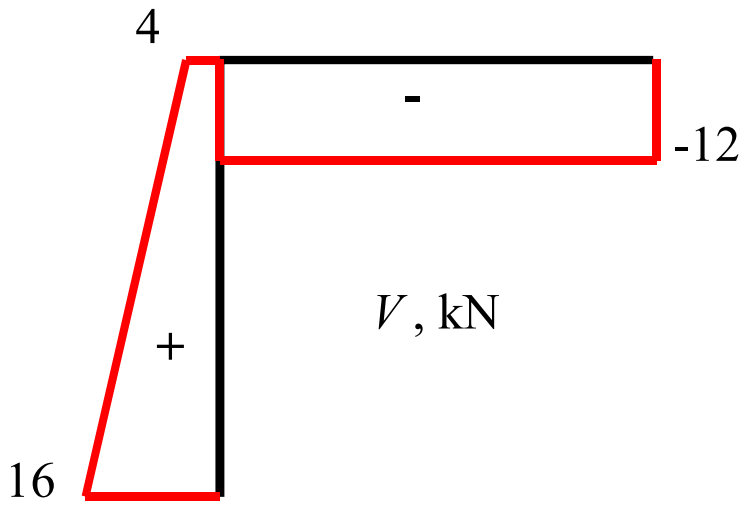


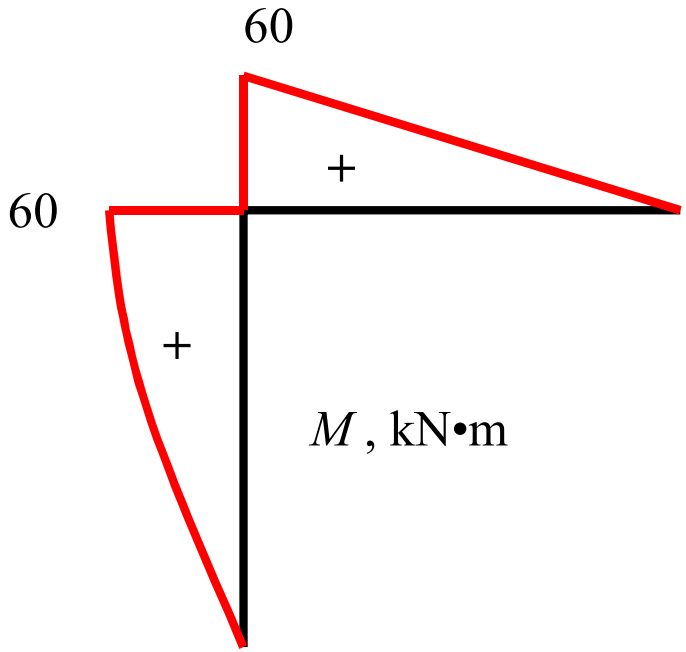
$$(1kN \bullet m)(\theta_C) = \int_0^L \frac{mM}{EI}\,dx = \frac{1}{1.5EI}\int_0^6 (0)(16x_1 - x_1^{\,2})dx_1 + \frac{1}{EI}\int_0^5 (1 - \frac{x_2}{5})(12x_2)dx_2$$

$$= 0 + \frac{1}{EI}\int_0^5 (12x_2 - \frac{12x_2^{\,2}}{5})dx_2$$

$$\theta_C = \frac{1}{EI}(\frac{12x_2^{\,2}}{2} - \frac{12x_2^{\,3}}{5\times 3})\Big|_0^5 = \frac{50}{EI} = \frac{50}{(200)(200)} = +0.00125\ \text{rad}\ ,$$

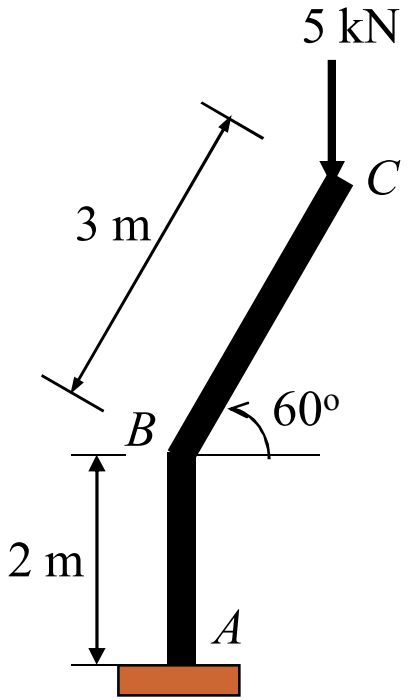

Δ_{CH}= = 28.87 mm
5 m
B
C
4 kN
θ_C = 0.00125 rad ,
12 kN
2 kN/m
6 m
A
16 kN
12 kN


4
-
-12
+
16
V , kN


60
+
60
+
M , kN•m

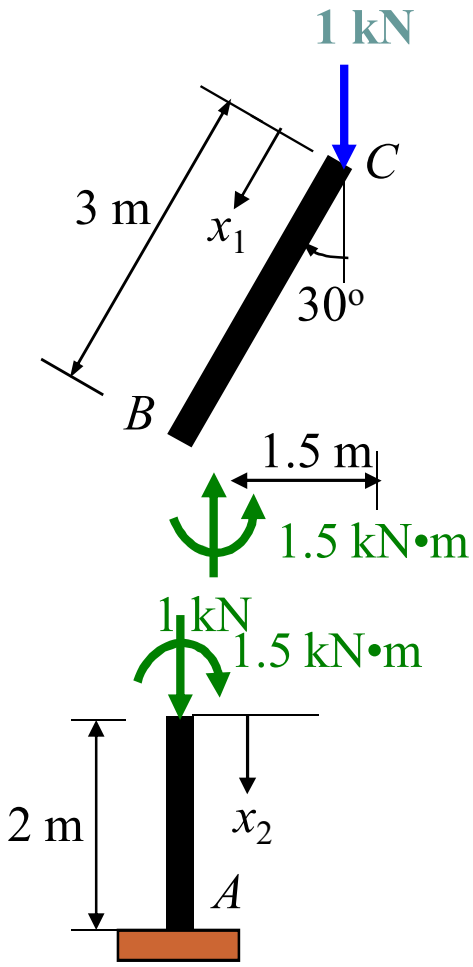
Example 8-22

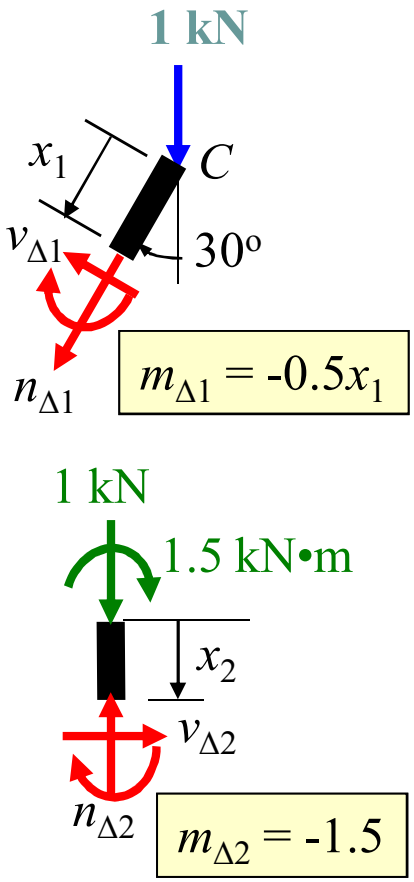
Determine the slope and the vertical displacement of point ***C*** on the frame. Take $E = 200$ GPa, $I = 15(10^6)\ \text{mm}^4$.



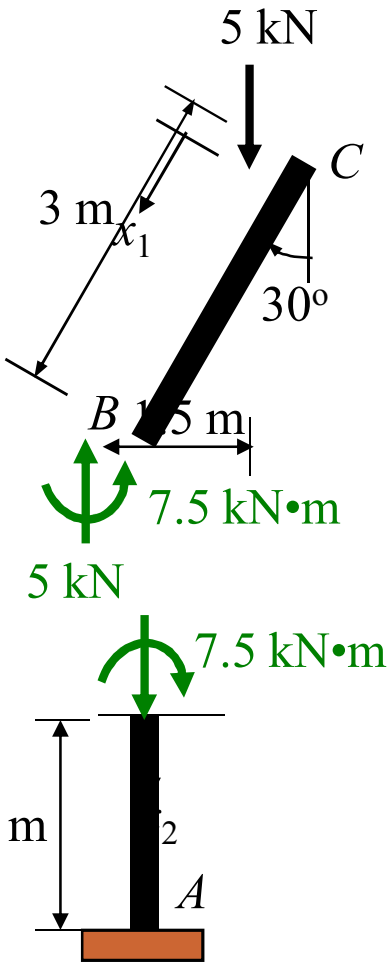
Displacement at C

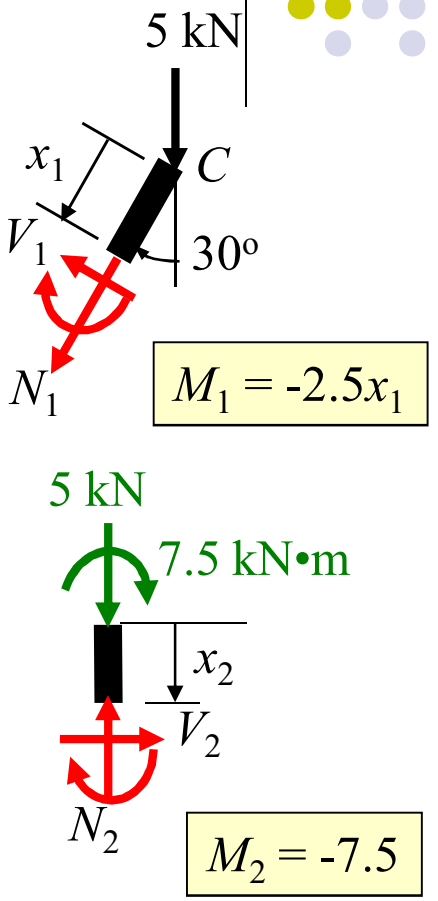
•Virtual Moment m_Δ





•Real Moment M





$$(1kN)(\Delta_{Cv}) = \int_0^L \frac{m_\Delta M}{EI}\,dx = \frac{1}{EI}\int_0^3 (-0.5x_1)(-2.5x_1)\,dx_1 + \frac{1}{EI}\int_0^2 (-1.5)(-7.5)\,dx_2$$

$$\Delta_{Cv} = \frac{1}{EI}\left(\frac{1.25x_1^{\,3}}{3}\right)\Big|_0^3 + \frac{1}{EI}\left(11.25x_2^{\,2}\right)\Big|_0^2 = \frac{33.75}{EI} = \frac{33.75}{(200)(15)} = 11.25\ \text{mm}\ \downarrow$$

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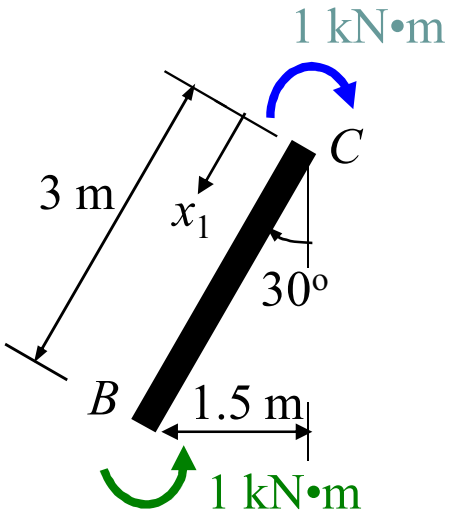
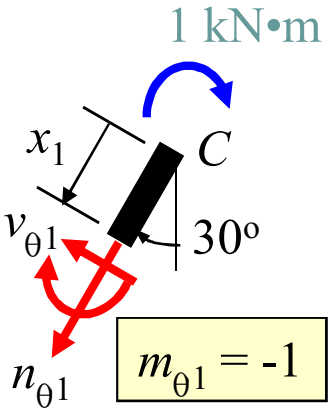
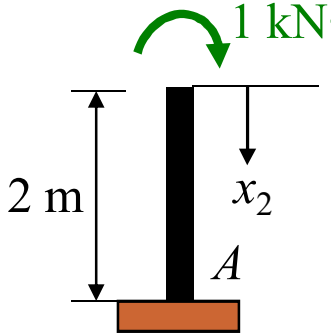
Slope at C

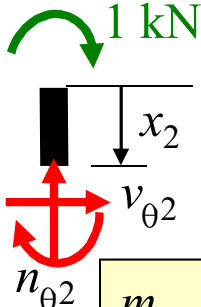
•*Virtual Moment* m_θ





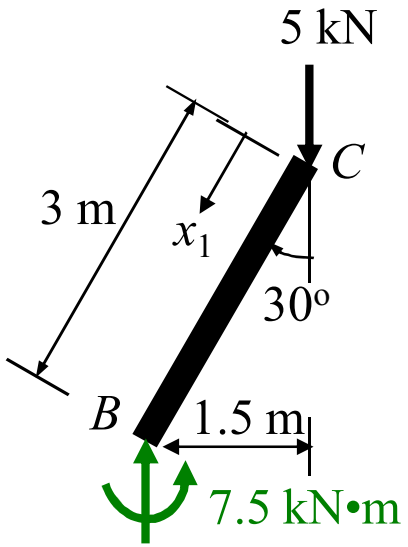
$m_{\theta 1} = -1$

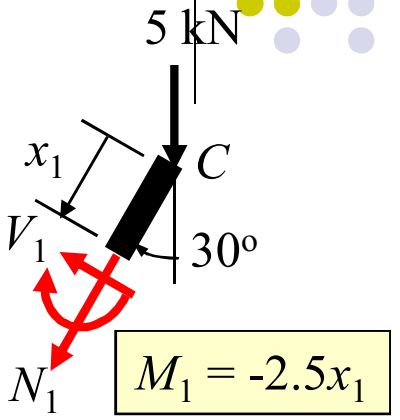




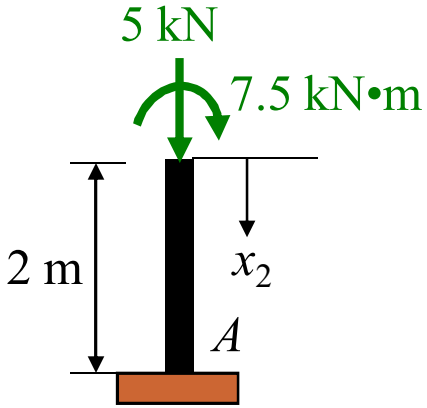
$m_{\theta 2} = -1$

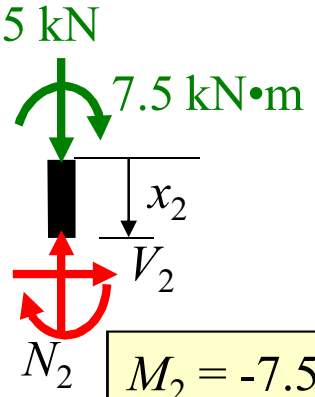
•*Real Moment M*





$M_1 = -2.5x_1$



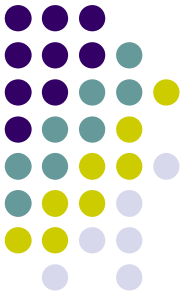


$M_2 = -7.5$

$$(1kN \bullet m)(\theta_C) = \int_0^L \frac{m_\theta M}{EI} dx = \frac{1}{EI}\int_0^3 (-1)(-2.5x_1)dx_1 + \frac{1}{EI}\int_0^2 (-1)(-7.5)dx_2$$

$$\theta_C = \frac{1}{EI}\left(\frac{2.5x_1^2}{2}\right)\Big|_0^3 + \frac{1}{EI}(7.5x_2)\Big|_0^2 = \frac{26.25}{EI} = \frac{26.25}{(200)(15)} = 0.00875 \text{ rad},$$

Virtual Strain Energy Caused by Axial Load, Shear, Torsion, and Temperature



- **Axial Load**

$$U_n = \int_0^L \frac{nN}{AE}\,dx$$

Where

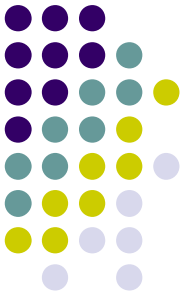
n = internal virtual axial load caused by the external virtual unit load
N = internal axial force in the member caused by the real loads
L = length of a member
A = cross-sectional area of a member
E = modulus of elasticity for the material

- **Bending**

$$U_b = \int_0^L \frac{mM}{EI}\,dx$$

Where

n = internal virtual moment cased by the external virtual unit load
M = internal moment in the member caused by the real loads
L = length of a member
E = modulus of elasticity for the material
I = moment of inertia of cross-sectional area, computed about the the neutral axis



- **Torsion**

$$U_t = \int_0^L \frac{tT}{GJ}\,dx$$

Where

t = internal virtual torque caused by the external virtual unit load
T = internal torque in the member caused by the real loads
G = shear modulus of elasticity for the material
J = polar moment of inertia for the cross section, $J = \pi c^4/2$, where c is the radius of the cross-sectional area

- **Shear**

$$U_s = \int_0^L K\left(\frac{vV}{GA}\right)dx$$

Where

v = internal virtual shear in the member, expressed as a function of x and caused by the external virtual unit load

V = internal shear in the member expressed as a function of x and caused by the real loads

K = form factor for the cross-sectional area:

$K = 1.2$ for rectangular cross sections

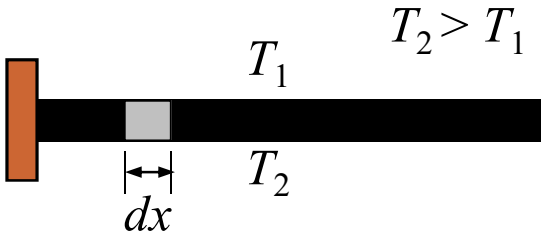
$K = 10/9$ for circular cross sections

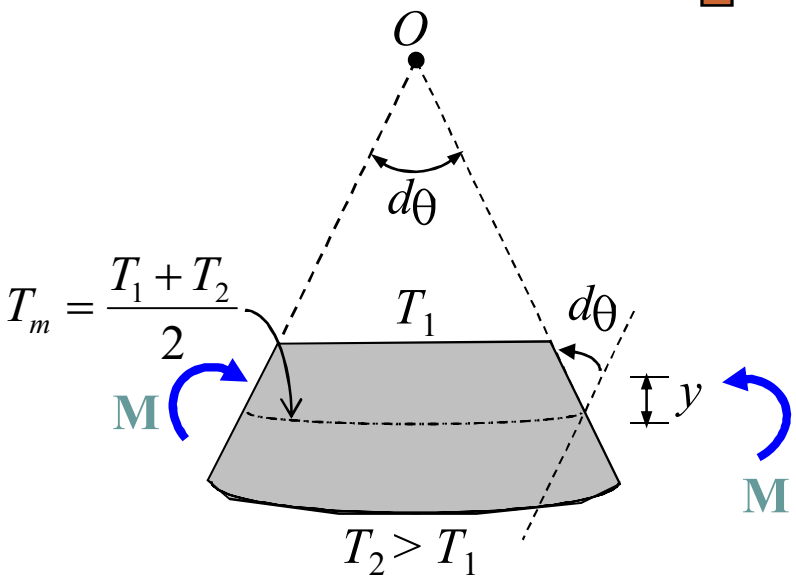
$K \approx 1$ for wide-flange and I-beams, where A is the area of the web

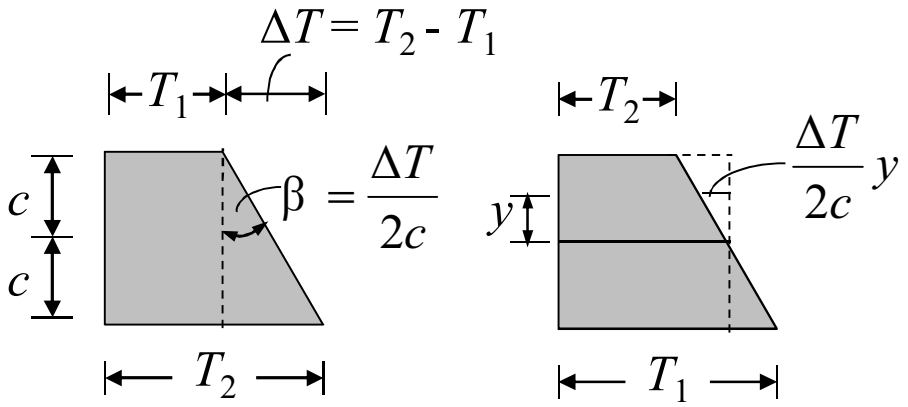
G = shear modulus of elasticity for the material

A = cross-sectional area of a member

- **Temperature**







$$(d\theta)y = \alpha(\frac{\Delta T}{2c}y)dx$$

$$(d\theta) = \alpha(\frac{\Delta T}{2c})dx$$

$$U_{temp} = \int m d\theta$$

$$U_{temp} = \int_0^L m(\alpha \frac{\Delta T}{2c})dx$$

Example 8-23

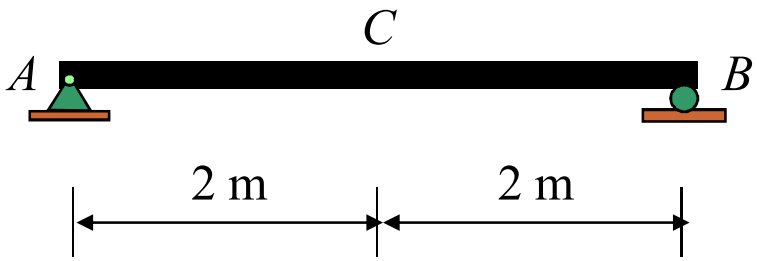
From the beam below Determine :

(a) If $P = 60$ kN is applied at the midspane C, what would be the displacement at point C. Due to shear and bending moment.

(b) If the temperature at the top surface of the beam is 55 ^{o}C , the temperature at the bottom surface is 30 ^{o}C and the room temperature is 25 ^{o}C. What would be the vertical displacement of the beam at its midpoint C and the the horizontal deflection of the beam at support B.

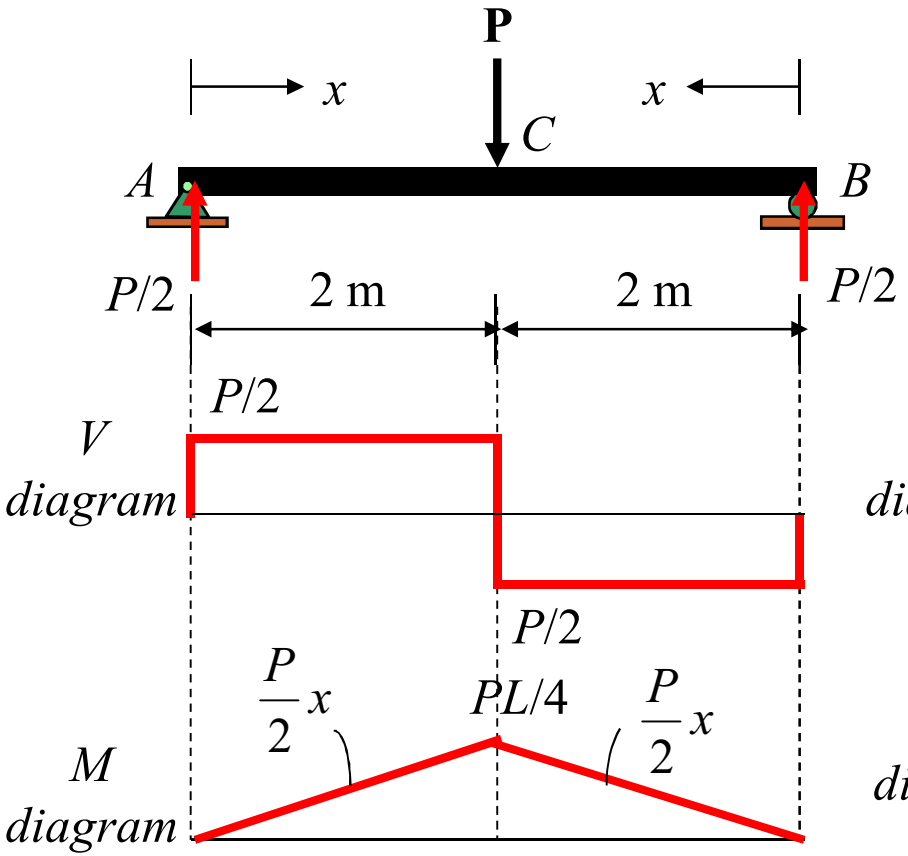
(c) if (a) and (b) are both accounted, what would be the vertical displacement of the beam at its midpoint C.

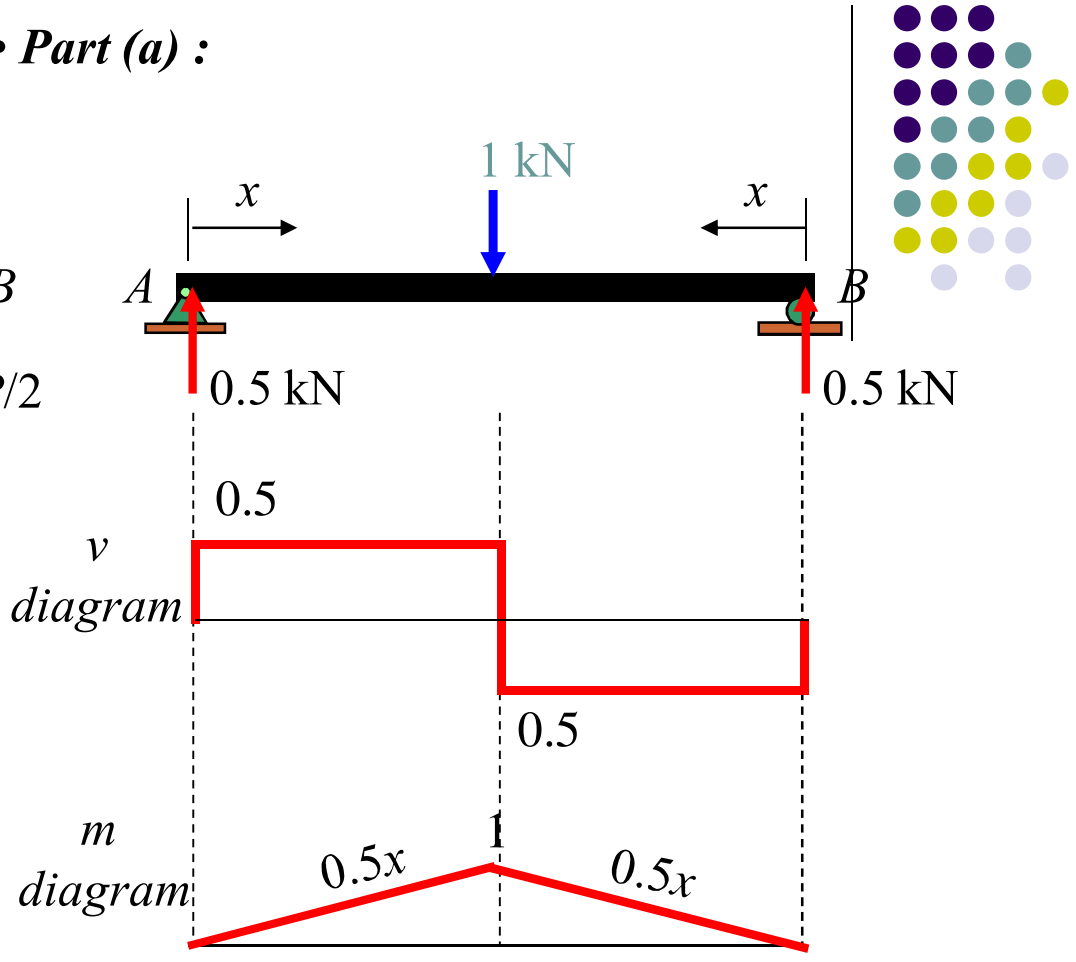
Take $\alpha = 12(10^{-6})/^{o}$C. $E = 200$ GPa, $G = 80$ GPa, $I = 200(10^{6})$ mm^4 and $A = 35(10^{3})$ mm^2. The cross-section area is rectangular.



SOLUTION

• ***Part (a) :***





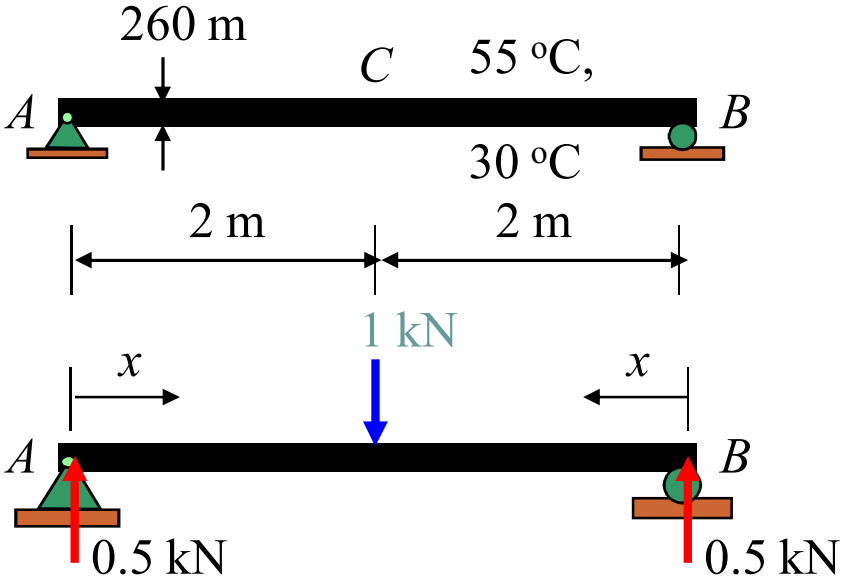
$$\Delta_{bending} = \int_0^L \frac{m_i M_i}{EI}\,dx = 2\int_0^{L/2} \left(\frac{x}{2}\right)\left(\frac{Px}{2}\right)\frac{dx}{EI} = \frac{2}{EI}\left(\frac{Px^3}{4\times 3}\right)\Big|_0^{L/2} = \frac{PL^3}{48EI} = \frac{60(4)^3}{48(200)(200)} = 2 \text{ mm}, \downarrow$$

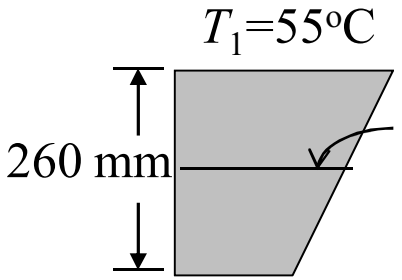
$$\Delta_{shear} = \int_0^L \frac{Kv_i V_i}{GA}\,dx = 2\int_0^{L/2} K\left(\frac{1}{2}\right)\left(\frac{P}{2}\right)\frac{dx}{GA} = \frac{KPx}{2GA}\Big|_0^{L/2} = \frac{KPL}{4GA} = \frac{1.2(60)(4)}{4(80)(35000)} = 0.026 \text{ mm}, \downarrow$$

$$\Delta_C = \Delta_{bending} + \Delta_{shear} = 2 + 0.026 = 2.03 \text{ mm}, \downarrow$$

SOLUTION

•***Part (b) : Vertical displacement at C***

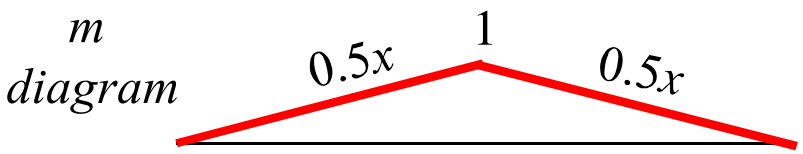




$$T_m = \frac{55+30}{2} = 42.5$$

$$T_{\text{room}} = 25\ ^{o}\text{C}\ ,$$

Temperature profile

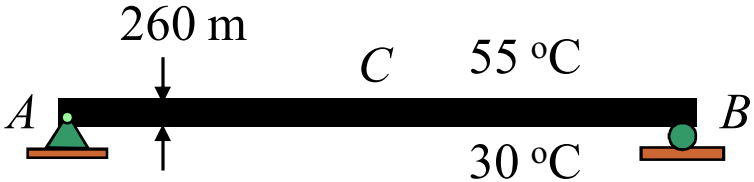


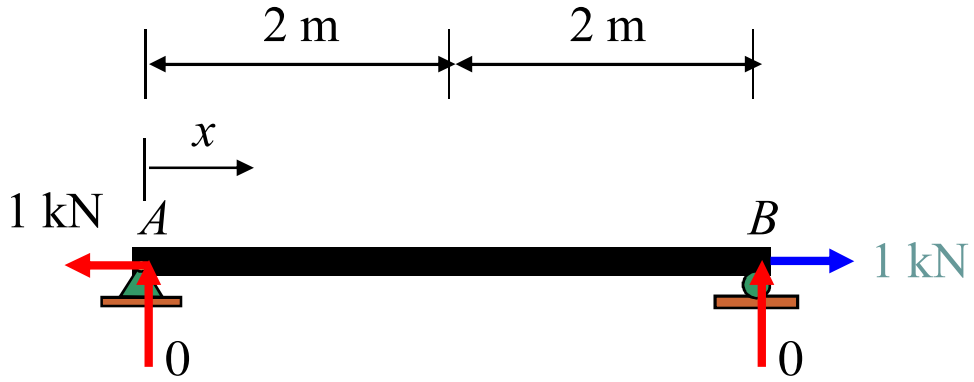
- Bending

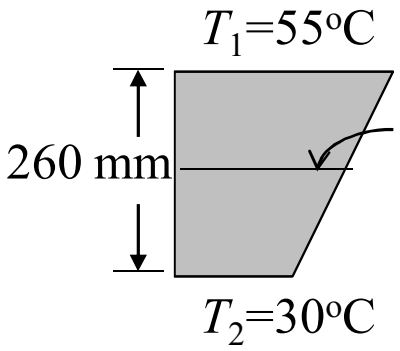
$$(1kN)(\Delta_{Cv}) = \int_0^L \frac{m\alpha(\Delta T)}{2c}dx \quad = 2\frac{\alpha(\Delta T)}{2c}\int_0^2 (0.5x)dx \quad = 2\frac{(12\times10^{-6})(-25)}{(260\times10^{-3})}\left(\frac{0.5x^2}{2}\right)\Bigg|_0^2$$

$$\Delta_{Cv} = \text{-}2.31\ \text{mm}\ ,\ \uparrow$$

• *Part (b) : Horizontal displacement at B*



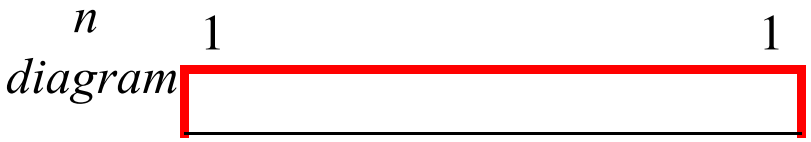




$$T_m = \frac{55+30}{2} = 42.5$$

$$T_{\text{room}} = 25\ ^{\circ}\text{C},$$

Temperature profile



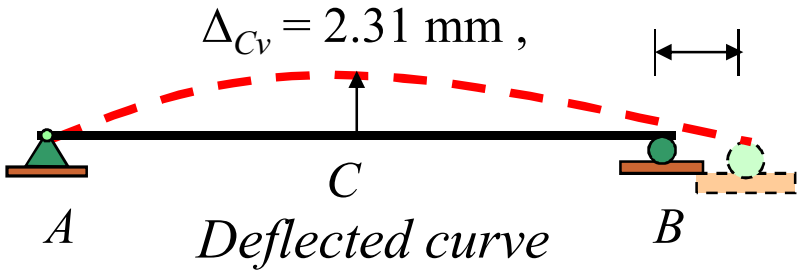
- Axial

$$(1kN)(\Delta_{BH}) = \int_0^L n\alpha(\Delta T)dx$$

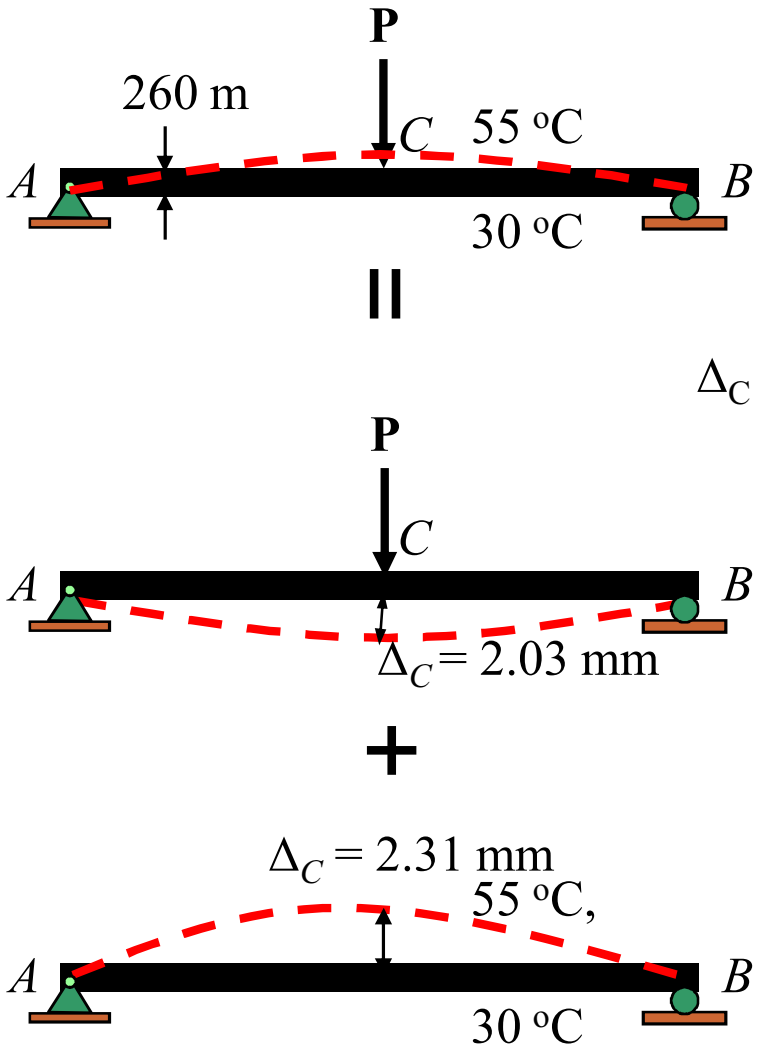
$$= \alpha(\Delta T)\int_0^4 (1)dx$$

$$= (12\times10^{-6})(42.5-25)(x)\Big|_0^4$$

$$\Delta_{BH} = 0.84 \text{ mm}, \longrightarrow$$



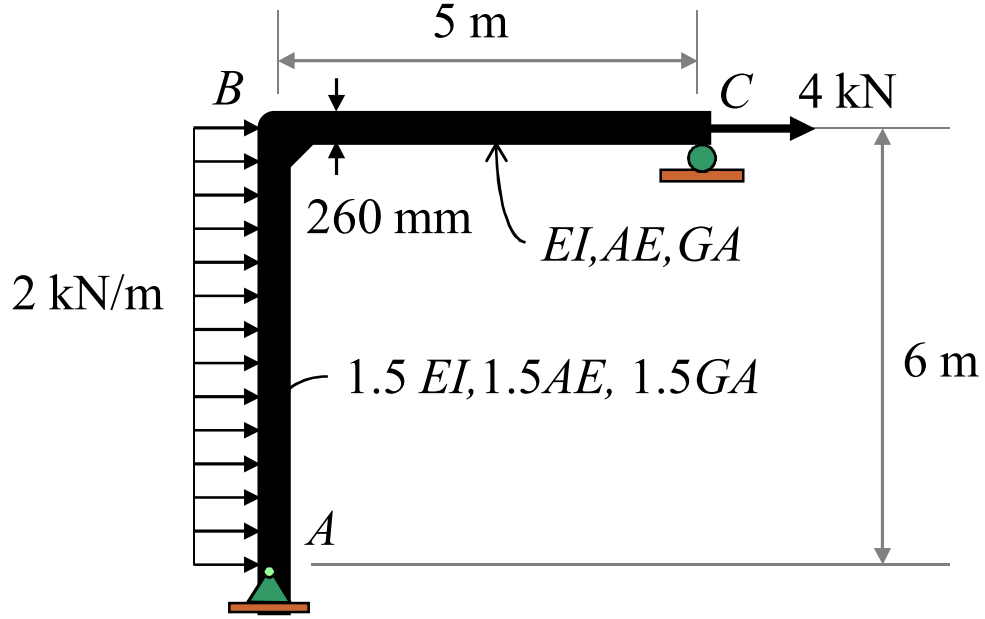
• ***Part (c) :***

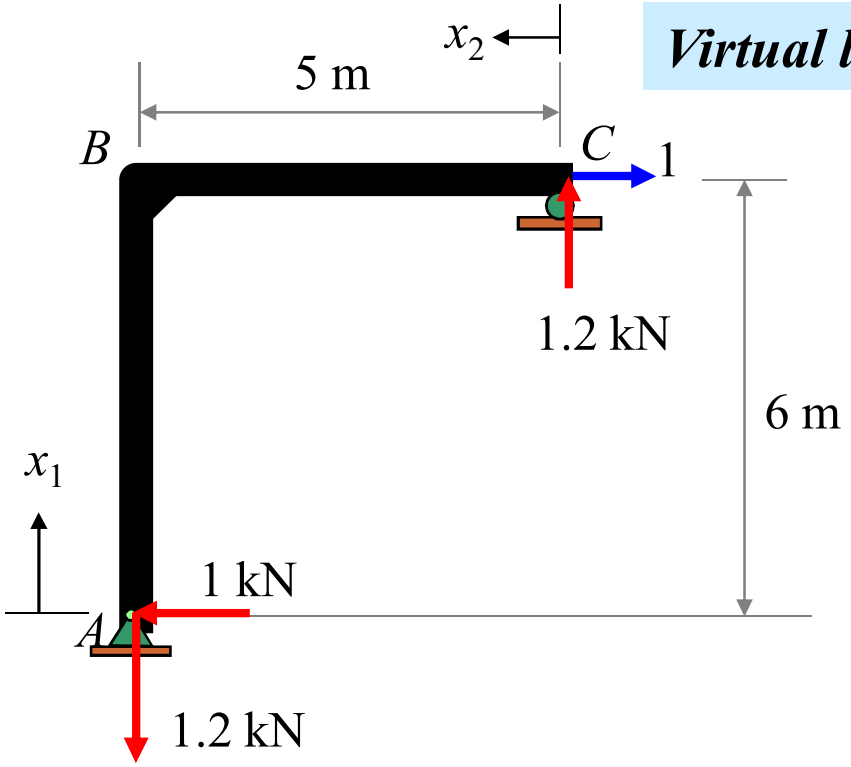


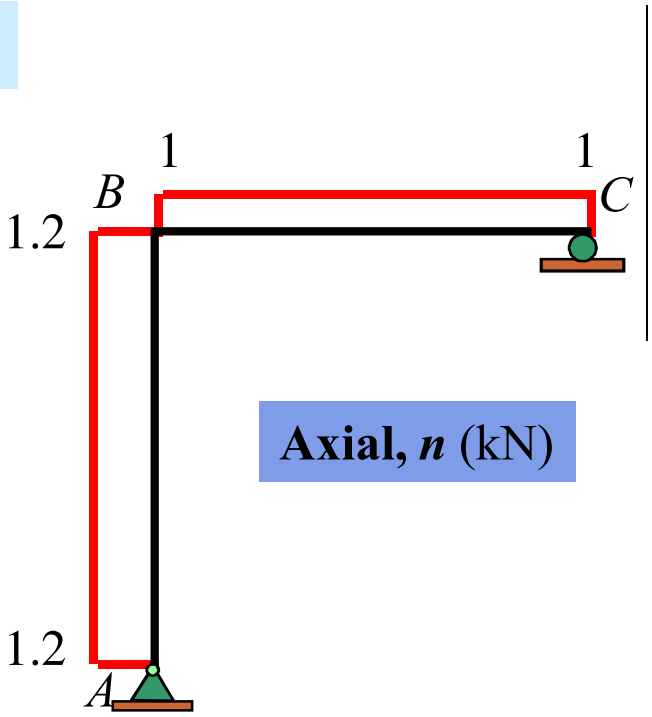
$\Delta_C = -2.03 + 2.31 = 0.28$ mm, ↑

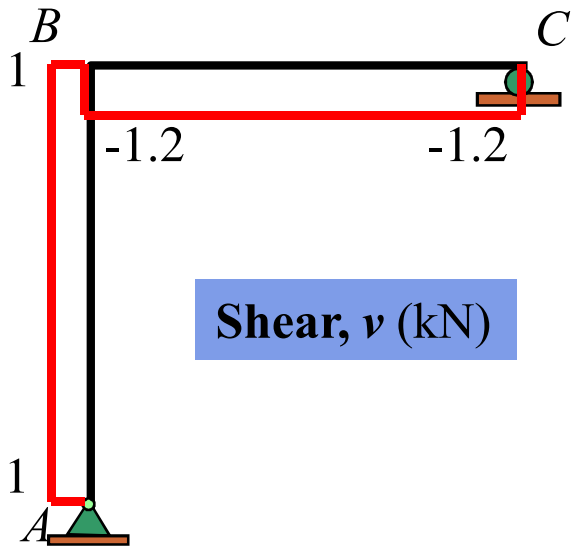
Example 8-24

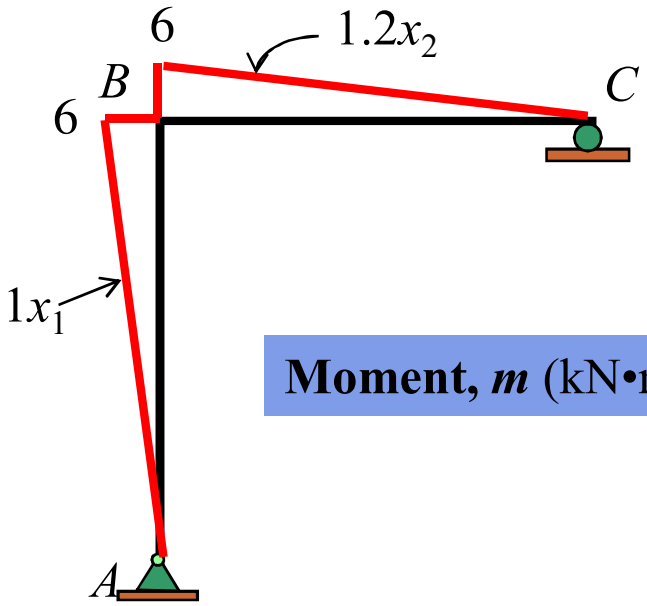
Determine the horizontal displacement of point ***C*** on the frame.If the temperature at top surface of member *BC* is 30 °C , the temperature at the bottom surface is 55 °C and the room temperature is 25 °C.Take $\alpha = 12(10^{-6})/^{o}C$, $E = 200$ GPa, $G = 80$ GPa, $I = 200(10^{6})\ mm^4$ and $A = 35(10^{3})\ mm^2$ for both members. The cross-section area is rectangular. Include the internal strain energy due to axial load and shear.

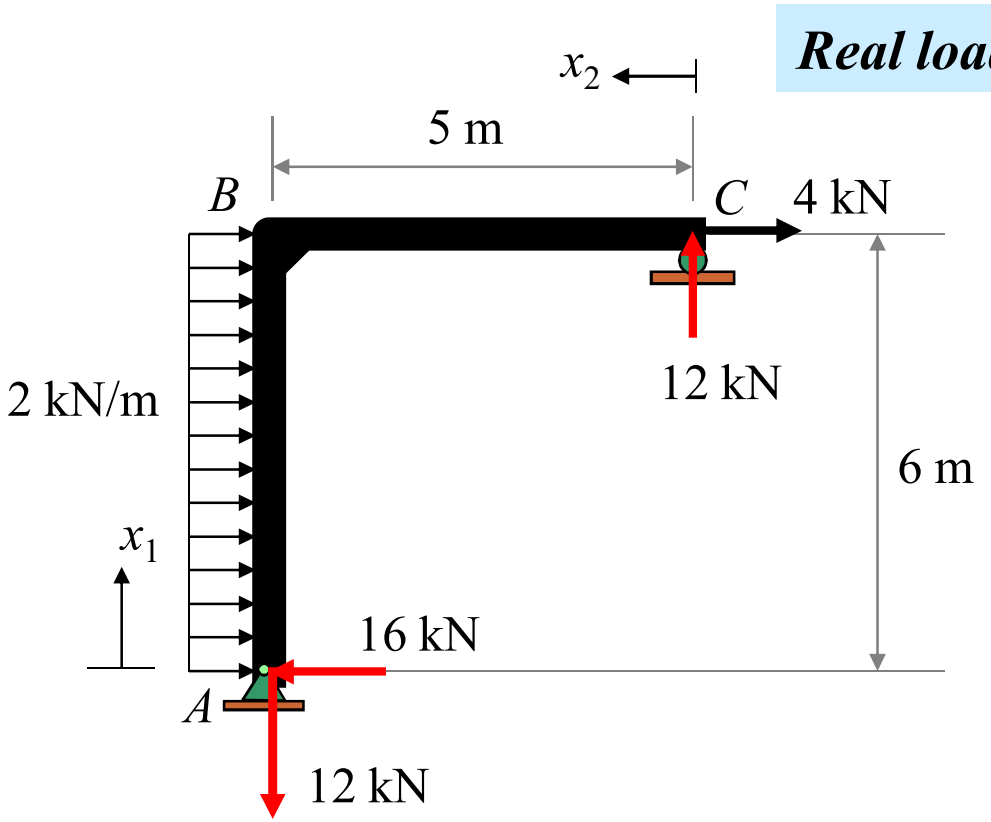


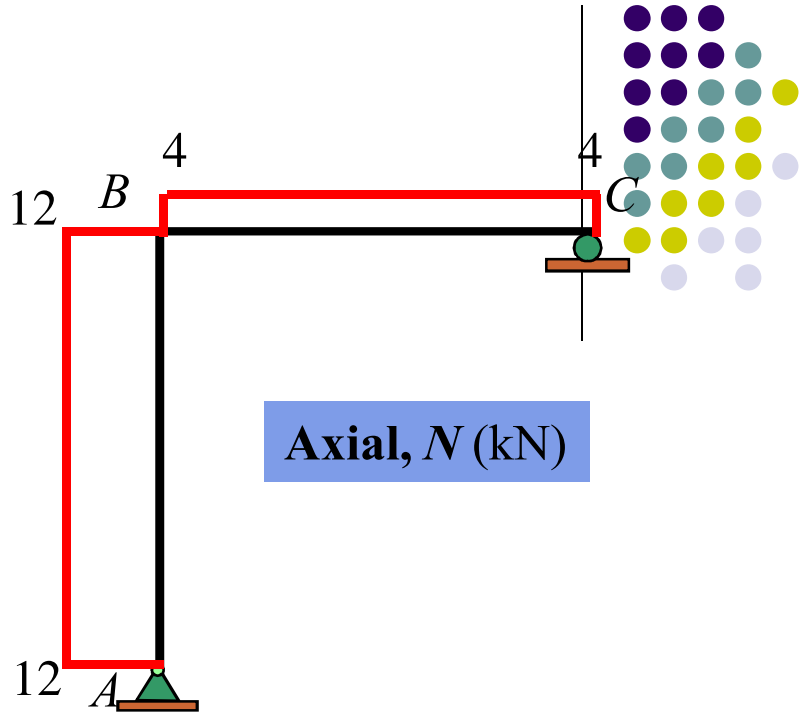

Virtual load
x_2
5 m
B
C
1
1.2 kN
6 m
x_1
1 kN
A
1.2 kN

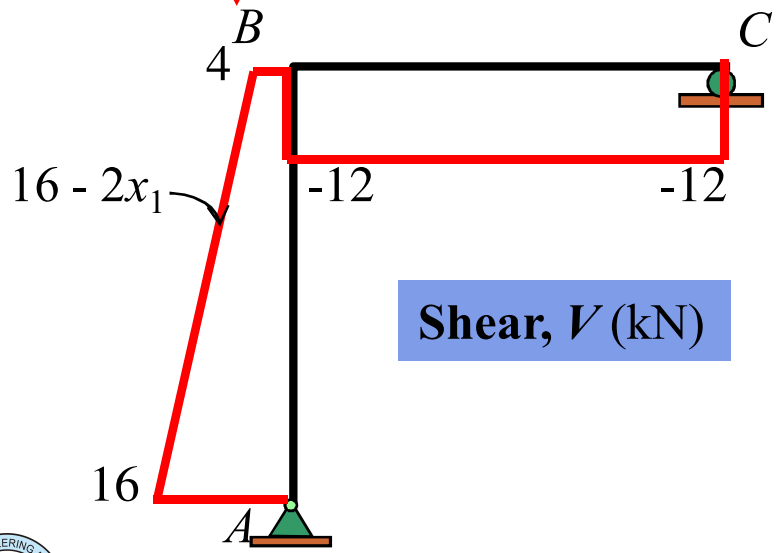

1
1
B
C
1.2
1.2
A
Axial, n (kN)

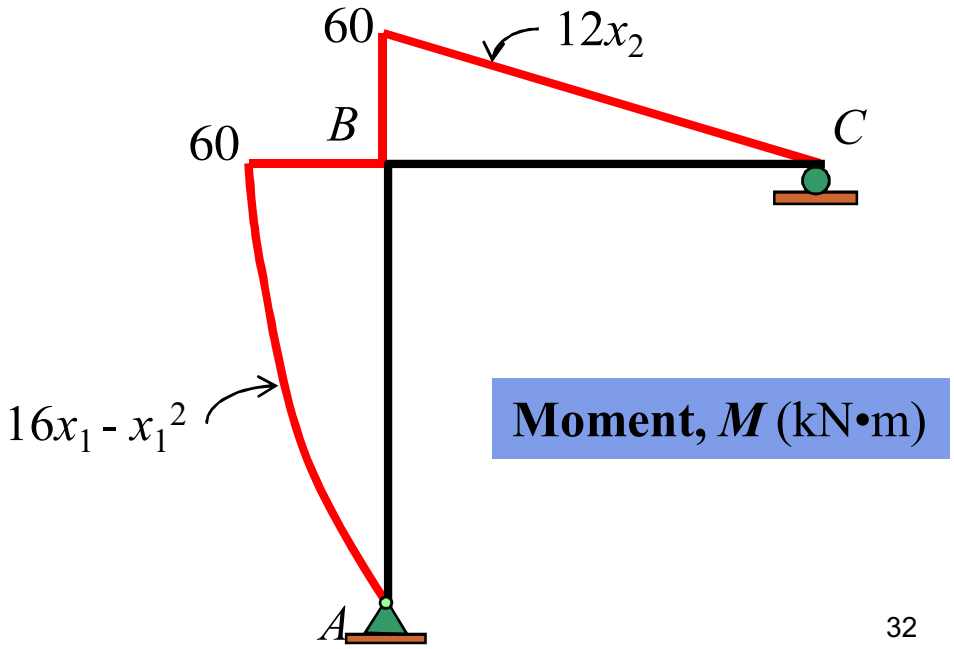

B
C
1
-1.2
-1.2
1
A
Shear, v (kN)


6
$1.2x_2$
B
C
6
$1x_1$
A
Moment, m (kN•m)



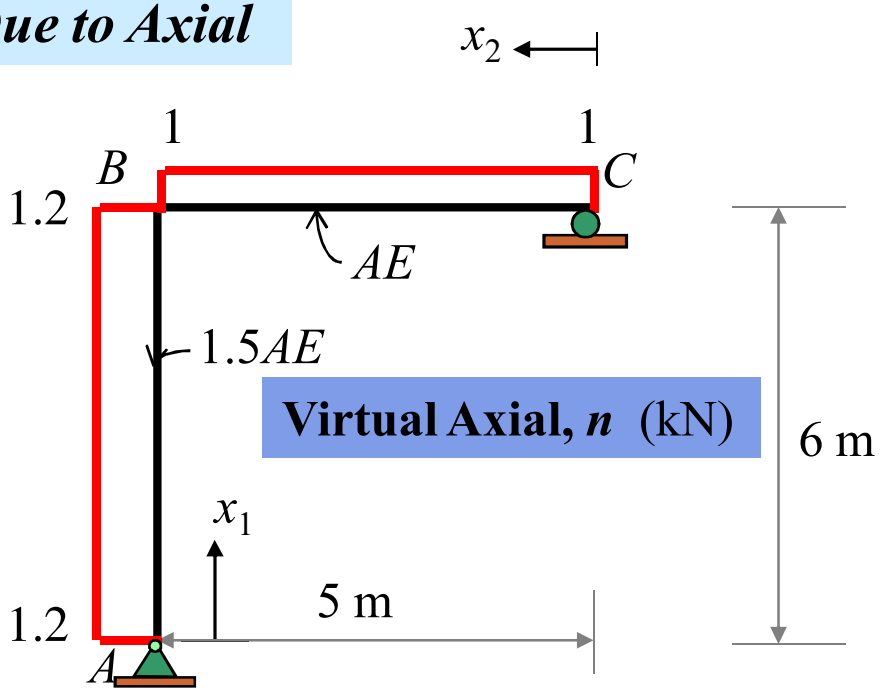


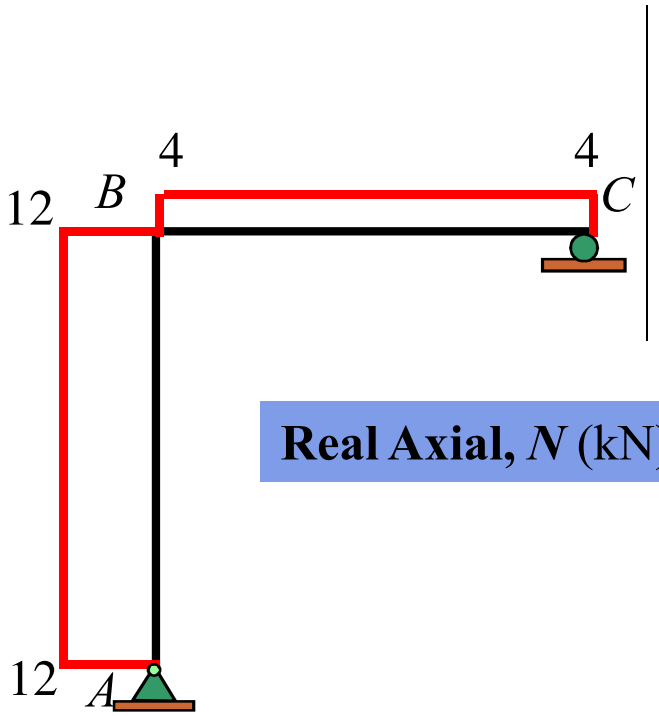




Department of Civil Engineering
University of Engineering and
Technology, Taxila, Pakistan

•*Due to Axial*





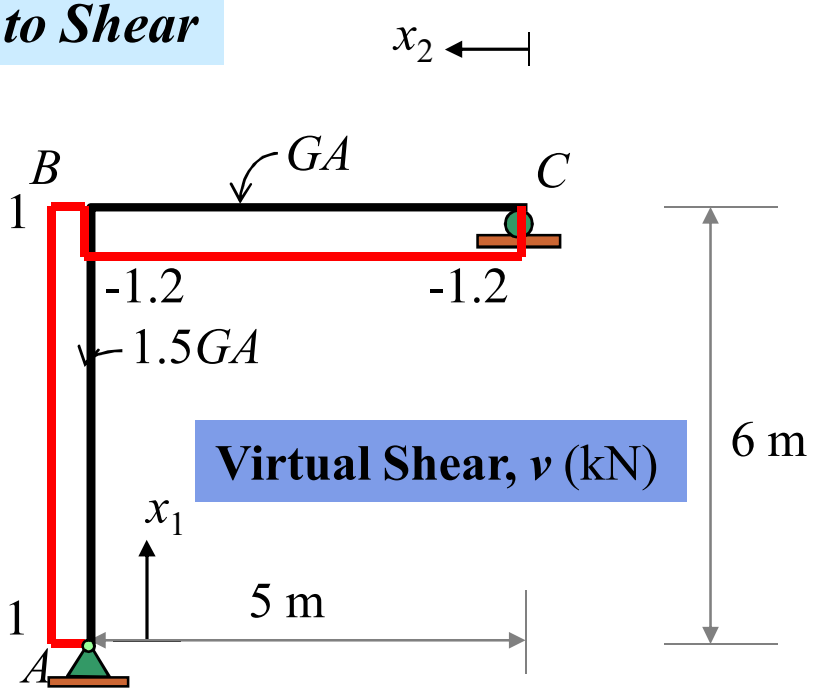
$$(1kN)(\Delta_{CH}) = \sum \frac{n_i N_i L_i}{A_i E_i}$$

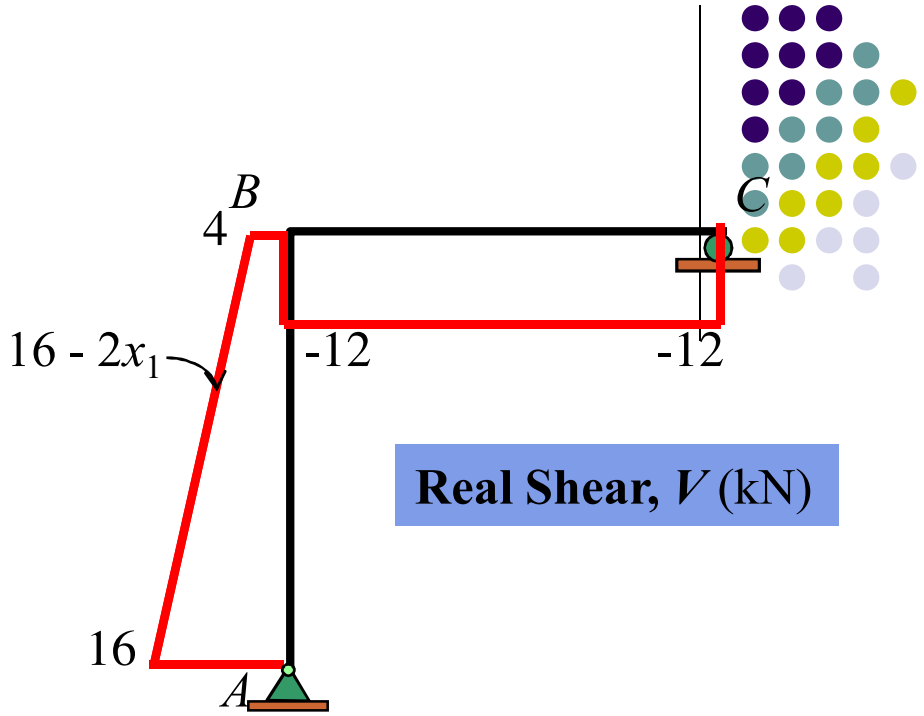
$$= \frac{(1.2)(12)(6)}{1.5AE} + \frac{(1)(4)(5)}{AE}$$

$$= \frac{77.6kN^2 \bullet m}{AE}$$

$$\Delta_{CH} = \frac{77.6kN \bullet m}{(35000\times10^{-6}m^2)(200\times10^{6}\frac{kN}{m^2})} = 1.109(10^{-5})\text{ m} = 0.0111\text{ mm}, \longrightarrow$$

•*Due to Shear*





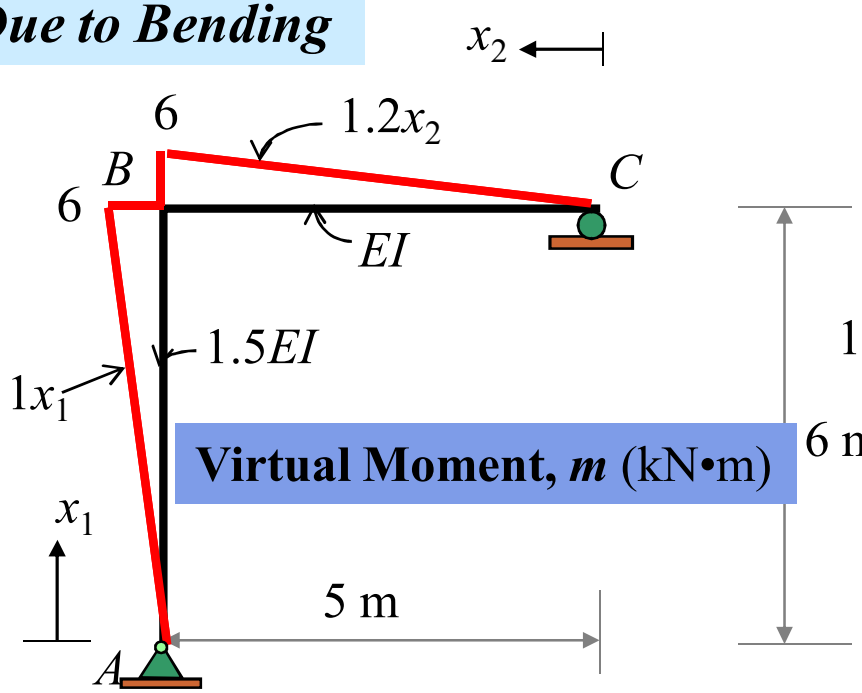
$$(1kN)(\Delta_{CH}) = \int_0^L K(\frac{vV}{GA})dx$$

$$= \int_0^6 1.2\frac{(1)(16-2x_1)}{1.5GA}dx_1 + \int_0^5 1.2\frac{(-1.2)(-12)}{GA}dx_2$$

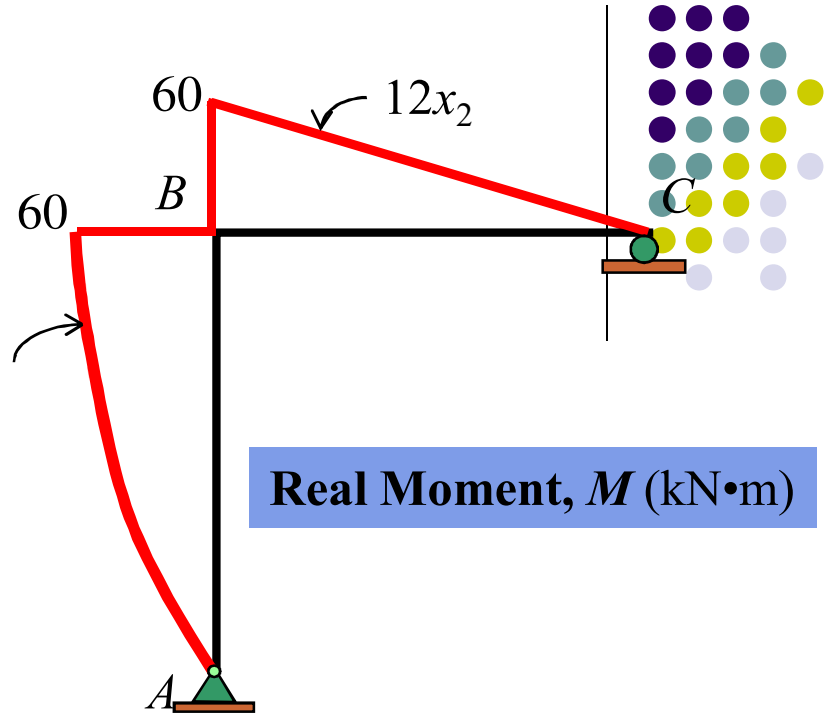
$$= (\frac{1.2}{1.5GA})(16x_1 - \frac{2x_1^{\,2}}{2})\Big|_0^6 + (\frac{1.2}{GA})(14.4x_2)\Big|_0^5 = \frac{134.4kN^2 \bullet m}{GA}$$

$$\Delta_{CH} = \frac{134.4kN \bullet m}{(80\times10^6\frac{kN}{m^2})(35000\times10^{-6}m^2)} = 4.8(10^{-5})\text{ m} = 0.048\text{ mm}, \longrightarrow$$

•*Due to Bending*



Virtual Moment, *m* (kN•m)



Real Moment, *M* (kN•m)

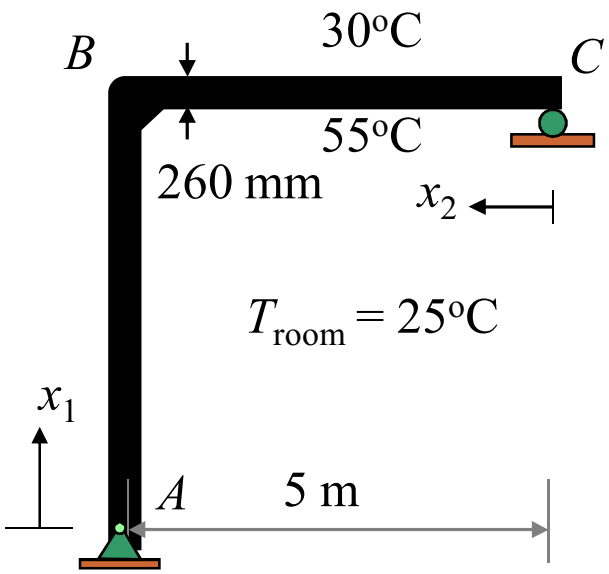
$$(1kN)(\Delta_{CH}) = \int_0^L \frac{mM}{EI}\,dx$$

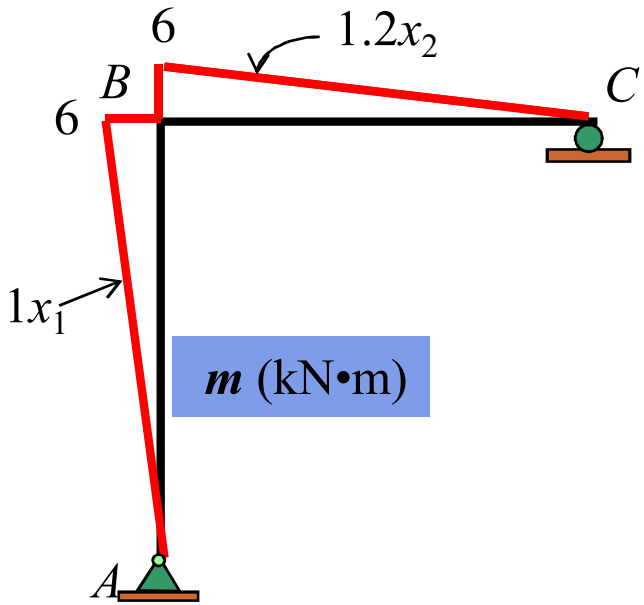
$$= \frac{1}{1.5EI}\int_0^6 (x_1)(16x_1 - x_1^{\,2})dx_1 + \frac{1}{EI}\int_0^5 (1.2x_2)(12x_2)dx_2$$

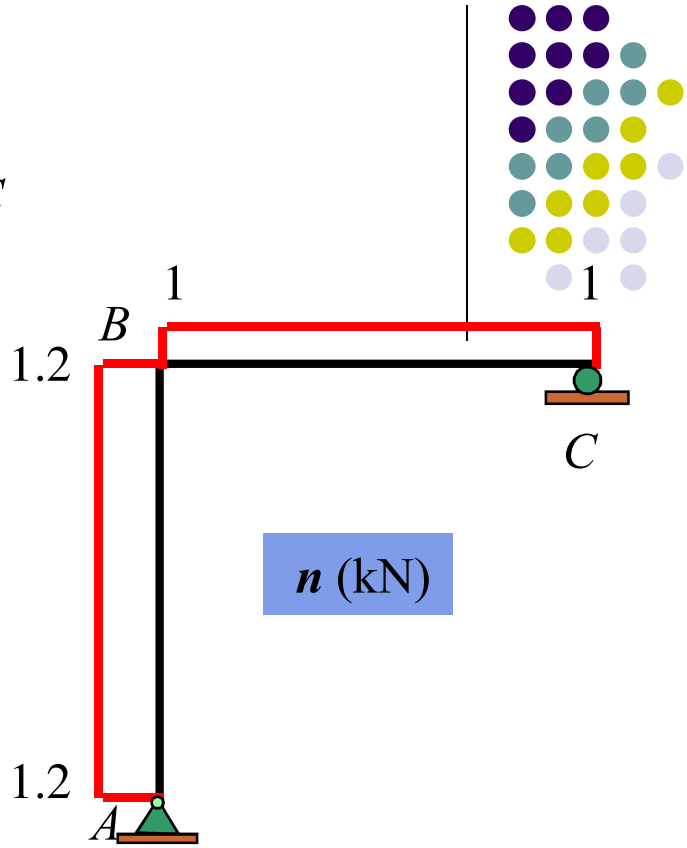
$$= \frac{1}{1.5EI}\left(\frac{16x_1^{\,3}}{3} - \frac{x_1^{\,4}}{4}\right)\Bigg|_0^6 + \frac{1}{EI}\left(\frac{14.4x_2^{\,3}}{3}\right)\Bigg|_0^5 = \frac{1152kN^2 \bullet m^3}{EI}$$

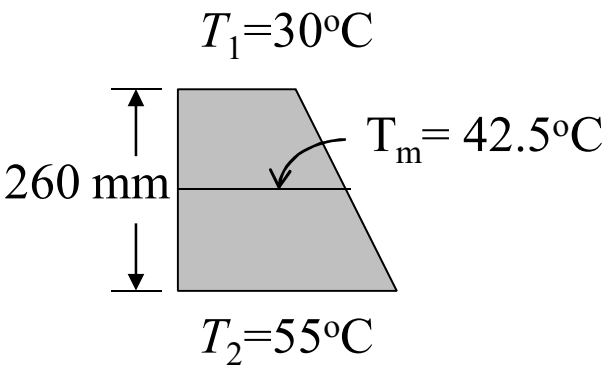
$$\Delta_{CH} = \frac{1152kN \bullet m^3}{(200\times10^6\,\frac{kN}{m^2})(200\times10^{-6}m^4)} = 0.0288\text{ m} = +28.8\text{ mm},\ \rightarrow$$

•*Due to Temperature*









Temperature profile

- *Bending*

$$(1kN)(\Delta_{CH}) = \int_0^L \frac{m\alpha(\Delta T)}{2c}dx = \int_0^5 \frac{(1.2x_2)(12\times10^{-6})(55-30)}{(260\times10^{-3})}dx^2$$

$$\Delta_{CH} = 0.0173 \text{ m} = 17.3 \text{ mm}, \longrightarrow$$

- *Axial*

$$(1kN)(\Delta_{CH}) = \int_0^L n\alpha(\Delta T)dx = \int_0^5 (1)(12\times10^{-6})(42.5-25)dx_2$$

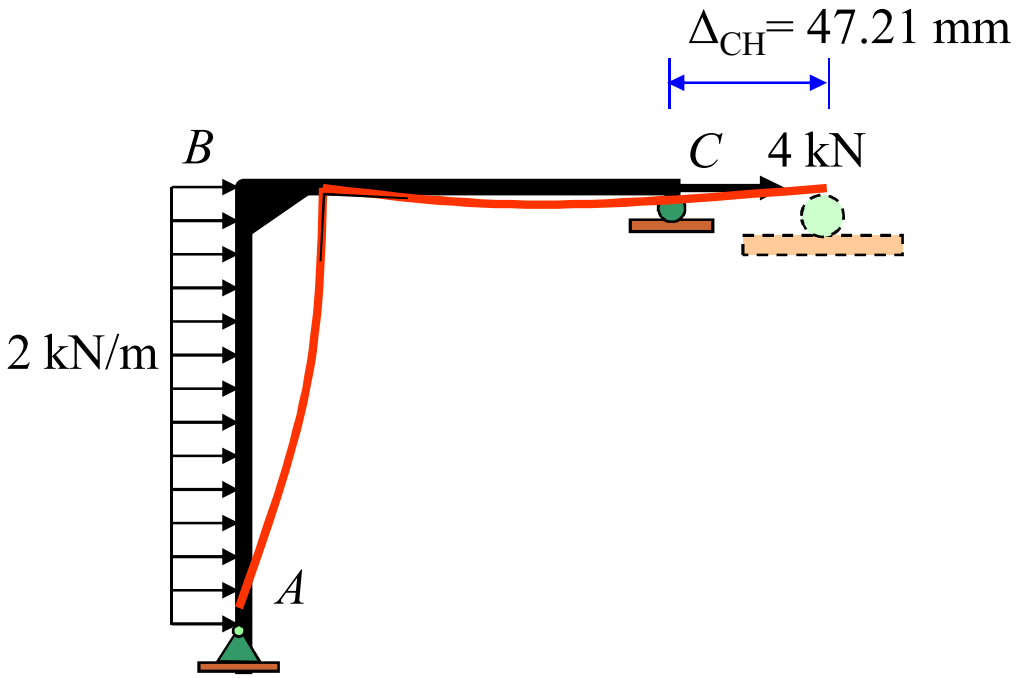
$$\Delta_{CH} = 0.00105 \text{ m} = 1.05 \text{ mm}, \longrightarrow$$

Department of Civil Engineering
University of Engineering and Technology, Taxila, Pakistan

•*Total Displacement*

$$(\Delta_{CH})_{Total} = (\Delta_{CH})_{Axial} + (\Delta_{CH})_{Shear} + (\Delta_{CH})_{Bending} + (\Delta_{CH})_{Temp}$$

$$= 0.01109 + 0.048 + 28.8 + (17.3 + 1.05) = 47.21 \text{ mm}$$



Thank You